

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2013
ROUND 1 COMPLEX NUMBERS (No Trig)

ANSWERS

A) (_____ , _____)

B) _____

C) (_____ , _____)

A) Given: $4i^{199} - 5i^{365} + 25i^{68} - (4i)^3 = x + yi$ for constants x and y
Compute the ordered pair (x, y) .

B) For k and c real, if $(2+i)^2 + k(1+2i) + c = (1-i)^3$, compute $\frac{k+c}{k-c}$.

C) The expanded product $(4-4i)^{100} \cdot (8+8i)^{60} = A^k$, where A and k are positive integers, k is as small as possible, and A is the largest possible power of 2 less than 1000. Compute the ordered pair (A, k) .

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2013
ROUND 2 ALGEBRA 1: ANYTHING**

ANSWERS

A) _____ ounces

B) \$ _____ . _____

C) _____

A) A 16 ounce solution of water and acid is 63% acid. Determine how much water (in ounces) must be added to the mixture to obtain a solution that is 36% acid.

B) Dick's stamps are valued at 46¢ each, while Joe's stamps are valued at 23¢ each. If Dick has three times as many stamps as Joe and the total value of all the stamps together is \$12.88, how much more are Dick's stamps worth than Joe's? Express your answer in dollars and cents.

C) Given the function of x denoted $f(x) = 5x - 3$

Compute the value of h for which $\frac{f(x+h) - f(x-h)}{h} = 3h - 8$.

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2013
ROUND 3 PLANE GEOMETRY: AREAS OF RECTILINEAR FIGURES**

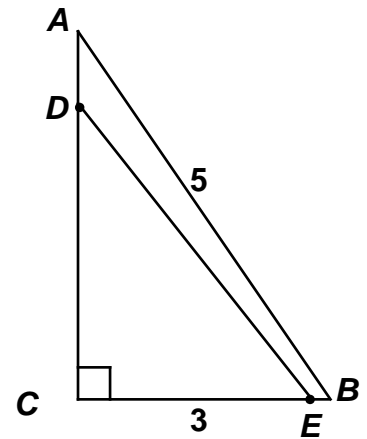
ANSWERS

A) _____ units²

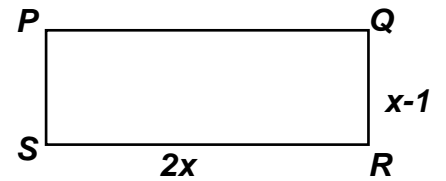
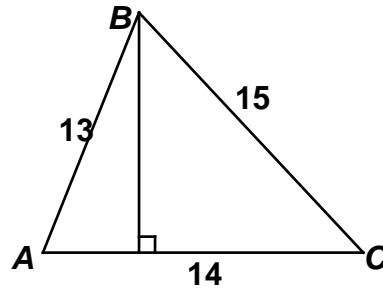
B) _____

C) _____ units²

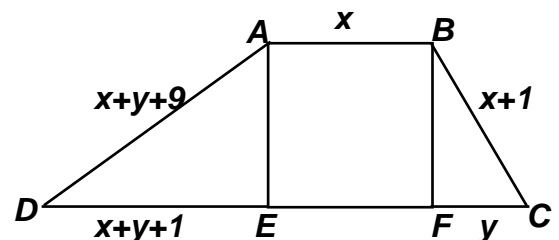
- A) In the diagram at the right, $AB = 5$, $BC = 3$, $DC = 7 \cdot AD$ and $CE = 11 \cdot BE$. Compute the area of $\triangle DEC$.



- B) Triangle ABC and rectangle $PQRS$ have the same area. Compute x .



- C) Trapezoid $ABCD$ is comprised of a square and two right triangles with sides as indicated in the diagram at the right. If the perimeter of trapezoid $ABCD$ is 152 units, compute the area of square $ABFE$.



**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2013
ROUND 4 ALG 1: FACTORING AND ITS APPLICATIONS**

ANSWERS

A) _____

B) _____

C) _____

A) Find all possible values of x for which $x^2 + 2x + 1 = 10000$.

B) Factor completely: $x(6x - 5y) + 4y(x - 4y) + y^2$

C) Solve for x .
$$\frac{4 - 7x - 2x^2}{5 - 11x + 2x^2} = 3 + \frac{38}{x - 12}$$

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2013
ROUND 5 TRIG: FUNCTIONS OF SPECIAL ANGLES**

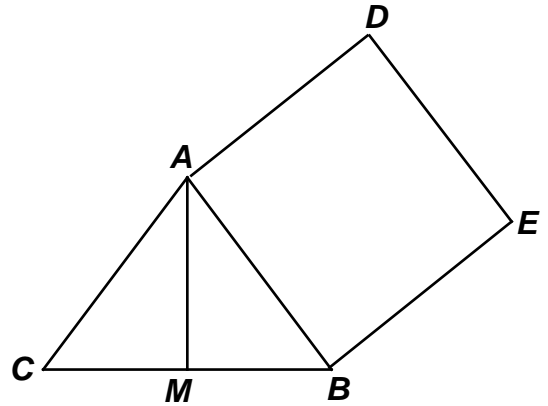
ANSWERS

A) _____

B) _____

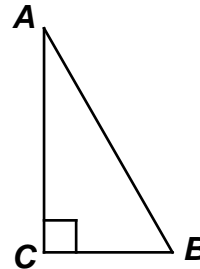
C) _____

- A) Given: $\triangle ABC$ is equilateral, M is the midpoint of \overline{BC} , and the area of square $ABED$ is 256. Compute AM .



- B) Given: $AB = 2BC$

Compute: $(\csc B \cot A \sin(750^\circ) \tan(-480^\circ) \sin(570^\circ))^5$



- C) Solve for θ over $0^\circ \leq \theta < 360^\circ$: $2 \sin \theta^\circ < \frac{\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ}{\cos 20^\circ \cos 40^\circ \cos 80^\circ \cos 120^\circ}$.

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2013
ROUND 6 PLANE GEOMETRY: ANGLES, TRIANGLES AND PARALLELS**

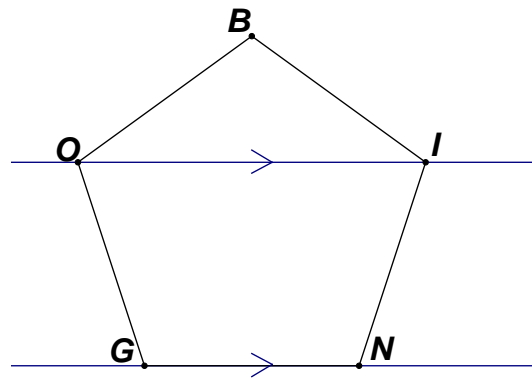
ANSWERS

A) _____

B) _____

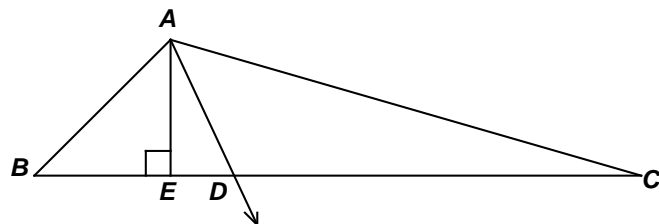
C) _____

- A) Given: Regular pentagon BINGO with $\overline{OI} \parallel \overline{GN}$
Compute $m\angle ING + m\angle BIO + m\angle OGI$.



- B) In kite $ABCD$, $m\angle BAC = m\angle DAC = (2x + 35)^\circ$ and $m\angle ABD = (80 - 3x)^\circ$.
Compute the value of x .

- C) In $\triangle ABC$, $\angle BAC$ is obtuse, \overline{AD} is an angle bisector of $\angle BAC$.
 $m\angle ABC = 30^\circ$, and $AC = 24$.
 $\overline{AE} \perp \overline{BC}$, $BE = 2DE$
Compute DC .



**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2013
ROUND 7 TEAM QUESTIONS
ANSWERS**

- A) _____ D) (_____ , _____ , _____ , _____)
 B) _____ E) _____
 C) _____ F) _____

A) For all integer values of k , $\left(\frac{4+4i}{5}\right)^{4k}$ is a real number.

Compute the minimum value of k for which $\left(\frac{4+4i}{5}\right)^{4k} > 8$.

Using $\log_{10} 2 = 0.3$ as an approximation is adequate for this computation.

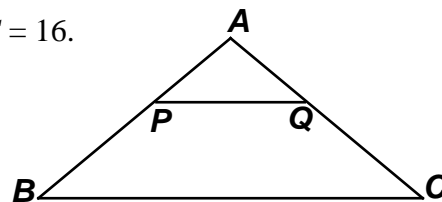
B) Find all positive two-digit integers N (in base 10) that satisfy both of the following statements:

- 1) The quotient of the integer divided by the positive difference of its digits is 21.
- 2) The sum of the product of the digits and the positive difference of the digits is 21.

C) Given: An isosceles triangle ABC with sides $AB = AC = 10$ and $BC = 16$.

The area of trapezoid $PQCB$ is 40.

Compute the height of the trapezoid.



D) If $x^{14} - x^8 - x^6 + 1$ is factored completely as a product of binomials and trinomials, where each lead coefficient is +1, the sum of these factors can be written in the form

$$Ax^4 + Bx^2 + Cx + D.$$

Determine the ordered quadruple (A, B, C, D) .

E) Given: For all x , $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$ (provided $x \neq 45^\circ + 180^\circ n$ for any integer n).

Determine all values of x over $0^\circ \leq x < 360^\circ$ which satisfy the following equation:

$$\frac{\cot 2x \cdot \cot x + 1}{\cot x - \cot 2x} = \tan 300^\circ$$

F) Suppose p and q are positive integers.

In regular polygon A with n sides, the ratio of an interior angle to an exterior angle is $p : q$, which is not necessarily in simplest form. In regular polygon B with m sides, the ratio of an exterior angle to an interior angle is $1 : p$. If $p = 11$, determine all possible ordered pairs (n, q) for which the ratio of an interior angle of A to an exterior angle of B is also an integer.

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2013 ANSWERS**

Round 1 Algebra 2: Complex Numbers (No Trig)

- A) (25, 55) B) 5 C) (32, 92)

Round 2 Algebra 1: Anything

- A) 12 oz B) \$9.20 C) 6

Round 3 Plane Geometry: Area of Rectilinear Figures

- A) $\frac{77}{16}$ ($4\frac{13}{16}$ or 4.8125) B) 7 C) 576

Round 4 Algebra 1: Factoring and its Applications

- A) 99, -101 B) $(3x-5y)(2x+3y)$ C) $-2, \frac{29}{4}$ (or 7.25)

Round 5 Trig: Functions of Special Angles

- A) $8\sqrt{3}$ B) $-\frac{9\sqrt{3}}{32}$ C) $210^\circ < \theta < 330^\circ$

Round 6 Plane Geometry: Angles, Triangles and Parallels

- A) 216 B) 25 C) $18\sqrt{3}$

Team Round

- A) 6 D) (1, 3, 4, 4)
B) 63 only E) $150^\circ, 330^\circ$
C) $6-\sqrt{6}$ F) (3, 22), (4, 11) and (24, 1)

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2013 SOLUTION KEY**

Round 1

A) The powers of i repeat after a cycle of 4 ($i, i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, \dots$).

$$4i^{199} - 5i^{365} + 25i^{68} - (4i)^3 = 4i^{4(49)+3} - 5i^{4(91)+1} + 25i^{4(17)} - (4i)^3 = -4i - 5i + 25 + 64i = 25 + 55i$$

Thus, $(x, y) = \underline{(25, 55)}$.

B) $(1-i)^3 = (1-i)^2(1-i) = (-2i)(1-i) = -2i - 2$

$$(2+i)^2 + k(1+2i) + c = 3 + 4i + k + 2ki + c = (3+k+c) + (2k+4)i$$

$$\text{Thus, } \begin{cases} 3+k+c = -2 \\ 2k+4 = -2 \end{cases} \Rightarrow (k, c) = (-3, -2) \Rightarrow \frac{-3-2}{-3+2} = \underline{5}.$$

C) $(4-4i)^{100} \cdot (8+8i)^{60} = 4^{100}(1-i)^{100} 8^{60}(1+i)^{60} = 2^{200}2^{180}((1-i)^2)^{50}((1+i)^2)^{30} = 2^{380}(-2i)^{50}(2i)^{30}$

$$2^{380}2^{50}(-1)2^{30}(-1) = 2^{460} \text{ and } 460 = 2^2 \cdot 5 \cdot 23$$

To minimize k , we must maximize A with the restriction that $A < 1000$.

This occurs when $A = 2^5$ and $k = 2^2 \cdot 23 = 92 \Rightarrow (A, k) = \underline{(32, 92)}$.

(Other factorizations of 460 produce a smaller A -value or an A -value exceeding 1000, e.g.

$$2^{2 \cdot 5} = 2^{10} = 1024 > 1000.)$$

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2013 SOLUTION KEY**

Round 2

A) $0.63(16) = 0.36(x+16) \Leftrightarrow 0.36x = 16(0.63 - 0.36)$

Multiplying through by 100, $36x = 16(63 - 36) = 16(27)$

Therefore, $x = \frac{16(27)}{36} = \frac{16(3)}{4} = \underline{\mathbf{12}}$.

B) Dick: $x \quad 46x + 23y = 1288$

Joe: $y \quad x = 3y$

$46(3y) + 23y = 1288 \Rightarrow 161y = 1288 \Rightarrow y = 8, x = 24$

The difference is $24(0.46) - 8(0.23) = 0.23(48 - 8) = 0.23(40) = \underline{\mathbf{\$9.20}}$.

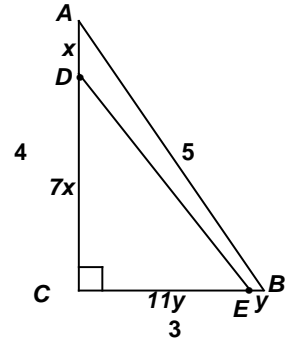
C) $f(x) = 5x - 3 \Rightarrow \frac{f(x+h) - f(x-h)}{h} = \frac{5(x+h) - 3 - (5(x-h) - 3)}{h} = \frac{10h}{h} = 3h - 8 \Rightarrow 3h = 18 \Rightarrow h = \underline{\mathbf{6}}$

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2013 SOLUTION KEY**

Round 3

A) $AC = 4, AD = x \Rightarrow 8x = 4 \Rightarrow DC = \frac{7}{2}, BE = y \Rightarrow 12y = 3 \Rightarrow CE = \frac{11}{4}$

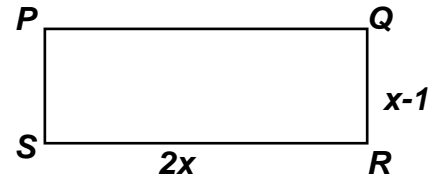
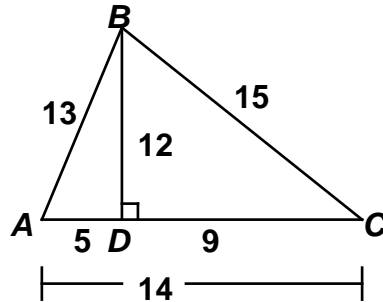
Thus, the required area is $\frac{1}{2} \cdot \frac{7}{2} \cdot \frac{11}{4} = \frac{77}{16}$ (or $4\frac{13}{16}$ or 4.8125).



B) Altitude \overline{BD} subdivides $\triangle ABC$ into two special triangles with sides 5-12-13 and 9-12-15

Thus, its area is $\frac{1}{2} \cdot 12 \cdot 14 = 84$ and, equating areas, we have

$$\begin{aligned} 2x(x-1) &= 84 \\ \Rightarrow x^2 - x - 42 &= 0 \\ \Rightarrow (x-7)(x+6) &= 0 \\ \Rightarrow x &= \underline{7} \end{aligned}$$



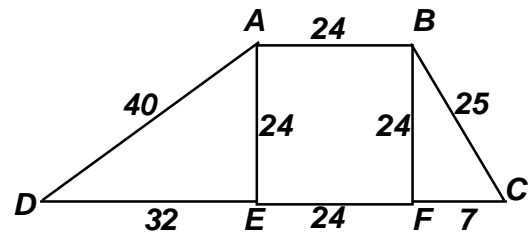
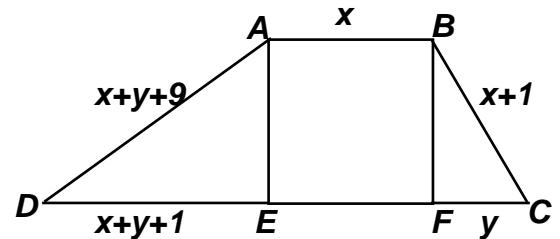
C) Per = $5x + 3y + 11 = 152 \Rightarrow 5x = 141 - 3y$

In $\triangle BCF$, $x^2 + y^2 = (x+1)^2 \Rightarrow 2x = y^2 - 1$

Substituting, $\frac{5}{2}(y^2 - 1) = 141 - 3y \Rightarrow 5y^2 - 5 = 282 - 6y \Rightarrow 5y^2 + 6y - 287 = 0$

$\Rightarrow (5y + 41)(y - 7) = 0 \Rightarrow y = 7 \Rightarrow x = 24$

$\Rightarrow \text{Area}(\text{square } ABFE) = \underline{576}$.



**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2013 SOLUTION KEY**

Round 4

A) $x^2 + 2x + 1 = 10000 \Leftrightarrow (x+1)^2 = 10^4 \Leftrightarrow x+1 = \pm 100 \Leftrightarrow x = \underline{\underline{99, -101}}$

Alternately, $x^2 + 2x + 1 = 10000 \Leftrightarrow x^2 + 2x - 9999 = 0$

Since 9999 is divisible by 99 [$99 \cdot 101$], we have factors which differ by 2.

$(x-99)(x+101) = 0 \Rightarrow x = \underline{\underline{99, -101}}$

B) $x(6x-5y) + 4y(x-4y) + y^2 \Leftrightarrow 6x^2 - xy - 15y^2 = \underline{\underline{(3x-5y)(2x+3y)}}$

C) $\frac{4-7x-2x^2}{5-11x+2x^2} = 3 + \frac{38}{x-12} \Leftrightarrow \frac{(4+x)(1-2x)}{(5-x)(1-2x)} = \frac{3x+2}{x-12}$ Note: $x \neq 5, \frac{1}{2}, 12$

Cancelling the common binomial factors on the left side and cross multiplying, we have

$(4+x)(x-12) = (5-x)(3x+2)$

$4x - 48 + x^2 - 12x = 15x + 10 - 3x^2 - 2x$

$4x^2 - 21x - 58 = 0$

$(4x-29)(x+2) = 0$

$x = \underline{\underline{-2, \frac{29}{4}}}$ (or 7.25)

FYI:

Check: $x = -2 \Rightarrow \frac{4+14-8}{5+22+8} = \frac{10}{35} = \frac{2}{7}, 3 + \frac{38}{-14} = 3 - \frac{19}{7} = \frac{2}{7}$

$$x = \frac{29}{4} \Rightarrow \frac{4 - \frac{203}{4} - \frac{841}{8}}{5 - \frac{19}{4} + \frac{841}{8}} = \frac{32 - 406 - 841}{40 - 638 + 841} = \frac{-1215}{243} = \frac{-5}{1}$$

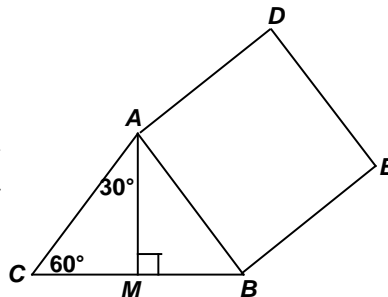
$$3 + \frac{38}{\frac{29}{4} - 12} = 3 + \frac{38(4)}{29 - 48} = 3 + \frac{38(4)}{-19} = 3 - 8 = \frac{-5}{1}$$

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2013 SOLUTION KEY**

Round 5

- A) \overline{AM} is also an altitude and an angle bisector, forcing $\triangle AMB$ to be $30^\circ - 60^\circ - 90^\circ$.

$$\text{Area}(ABED) = 256 \Rightarrow AB = 16 \Rightarrow MB = 8 \Rightarrow AM = \underline{8\sqrt{3}}$$



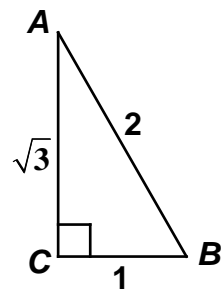
- B) $AB = 2BC \Rightarrow \triangle ABC$ is $30^\circ - 60^\circ - 90^\circ$ or let $AB = 2$ and $BC = 1$, solve for AC ($= \sqrt{3}$) and use SOHCAHTOA to determine values for $\csc B$ and $\cot A$.

Since adding (or subtracting) multiples of 360° to (or from) the arguments of any trig function gives us a coterminal angle, the value of the trig function is unchanged.

$$(\csc B \cot A \sin(750^\circ) \tan(-480^\circ) \sin(570^\circ))^5 =$$

$$\left(\frac{2}{\sqrt{3}} \cdot \sqrt{3} \sin 30^\circ \tan 240^\circ \sin 210^\circ \right)^5$$

$$\left(2 \cdot \frac{1}{2} \cdot \sqrt{3} \cdot \left(-\frac{1}{2} \right) \right)^5 = \left(-\frac{\sqrt{3}}{2} \right)^5 = \underline{-\frac{9}{32}\sqrt{3}}$$

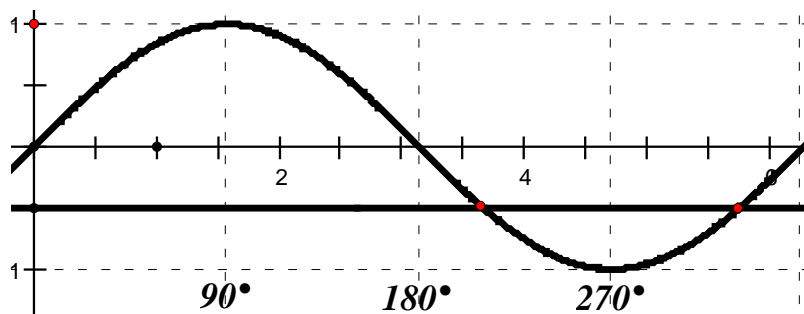


- C) Note that $\sin 10^\circ = \cos 80^\circ$, $\sin 30^\circ = \cos 60^\circ$, $\sin 50^\circ = \cos 40^\circ$

$$\text{So } \frac{\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ}{\cos 20^\circ \cos 40^\circ \cos 80^\circ \cos 120^\circ} = \frac{\cancel{\sin 10^\circ} \cdot \cancel{\sin 50^\circ} \cdot \cancel{\sin 70^\circ} \cdot \sin 30^\circ}{\cancel{\cos 80^\circ} \cdot \cancel{\cos 40^\circ} \cdot \cancel{\cos 20^\circ} \cdot \cos 120^\circ} = \frac{\cos 60^\circ}{-\cos 60^\circ} = -1.$$

$$\sin 30^\circ = \frac{1}{2} \Rightarrow \sin 150^\circ = \frac{1}{2}, \sin 210^\circ = -\frac{1}{2}, \sin 330^\circ = -\frac{1}{2}$$

Therefore, the original equation is equivalent to $\sin \theta < -\frac{1}{2}$. From the graph of the sine function over the specified interval, we have a solution set of $\underline{210^\circ < \theta < 330^\circ}$.



**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2013 SOLUTION KEY**

Round 6

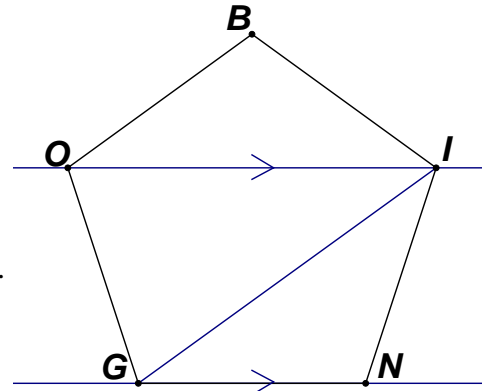
A) Ignoring the parallels, $m\angle ING = \frac{180 \cdot 3}{5} = 108^\circ$.

As base angles of isosceles triangles BIO and ING ,

$$m\angle BIO = m\angle NGI = \frac{180 - 108}{2} = 36^\circ$$

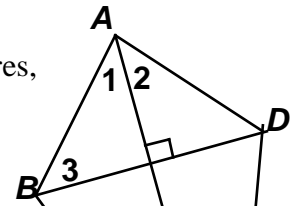
$$\Rightarrow m\angle OGI = 108 - 36 = 72^\circ$$

Therefore, $m\angle ING + m\angle BIO + m\angle OGI = 108 + 36 + 72 = \underline{216}$.



B) Since angles BAC (1) and DAC (2) have the same vertex and equal measures, they must be reflection angles across the main diagonal. To insure that the diagonals are perpendicular, angles BAC and ABD (3) must be complementary. Therefore,

$$(2x + 35) + (80 - 3x) = 90 \Rightarrow 115 - x = 90 \Rightarrow x = \underline{25}$$



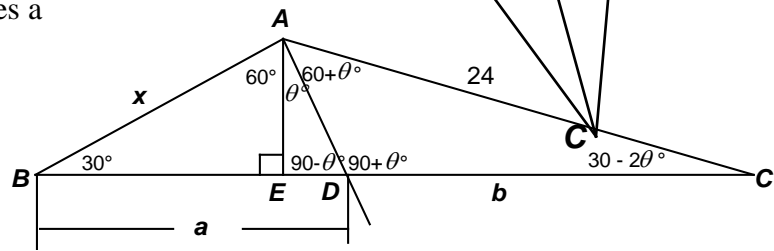
C) Dropping a perpendicular from A to \overline{BC} creates a 30-60-90 right triangle.

$$AE = \frac{x}{2}, BE = \frac{x\sqrt{3}}{2} \text{ and } DE = \frac{x\sqrt{3}}{4}$$

$$\Rightarrow a = BD = \frac{x\sqrt{3}}{2} + \frac{x\sqrt{3}}{4} = \frac{3x\sqrt{3}}{4}$$

Applying the angle bisector theorem,

$$\frac{a}{x} = \frac{b}{24} \Leftrightarrow \frac{\frac{3x\sqrt{3}}{4}}{x} = \frac{b}{24} \Leftrightarrow b = \underline{18\sqrt{3}}$$



FYI: Without the fact that $BE = 2DE$, a unique value for $m\angle CAD$ could not have been determined. In $\triangle ACD$, we know two side lengths (AC and DC , but not the included angle). In fact, the only invariant in $\triangle CAD$ is DC . Changing x changes the length of AD and the measures of all three angles in $\triangle CAD$, but the length of DC remains the same.

We know that $\theta < 30^\circ$ (to insure that $\angle BAC$ is obtuse). In this problem, in $\triangle ADE$, there is a unique acute angle whose tangent has this value. Its exact designation is $\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and its

approximate value is 40.9° , so with this additional condition ($BE = 2DE$),

$\triangle CAD$ is uniquely determined.

An easy way to confirm this approximation would be to draw a right triangle with legs of length 2 and $\sqrt{3}$; then measure the smaller acute angle with a protractor!

**MASSACHUSETTS MATHEMATICS LEAGUE
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Team Round

$$A) \left(\frac{4+4i}{5} \right)^{4k} = \frac{4^{4k} \left((1+i)^2 \right)^{2k}}{5^{4k}} = \frac{2^{8k} (2i)^{2k}}{5^{4k}} = \frac{2^{8k} 2^{2k} (i^2)^k}{5^{4k}} = \frac{(-1)^k 2^{10k}}{5^{4k}}$$

Thus, k must be even to insure that the quotient is positive.

For even values of k , we ignore $(-1)^k$.

We require that $\frac{2^{10k}}{5^{4k}} > 8 \Leftrightarrow 2^{10k} > 8(5^{4k})$. Taking \log_{10} of both sides, we have

$$10k(\log_{10} 2) > 3\log_{10} 2 + 4k(\log_{10} 5)$$

$$\log_{10} 2 \approx 0.3 \Rightarrow \log_{10} 5 = \log_{10} \left(\frac{10}{2} \right) = \log_{10} 10 - \log_{10} 2 = 1 - \log_{10} 2 \Rightarrow \log_{10} 5 \approx 0.7$$

$$\text{Substituting, } 10k(\log_{10} 2) > 3\log_{10} 2 + 4k(\log_{10} 5) \Rightarrow 3k > 0.9 + 2.8k \Rightarrow 2k > 9 \Rightarrow k > 4.5$$

Therefore, the minimum value of k is **6**.

Alternate Solution:

$$\left(\frac{4+4i}{5} \right)^{4k} = \left(\left(\frac{4(1+i)}{5} \right)^4 \right)^k = \left(\frac{2^8 \left((1+i)^2 \right)^2}{5^4} \right)^k = \left(\frac{2^8 (2i)^2}{5^4} \right)^k = \left(\frac{-1024}{625} \right)^k$$

Clearly k must be even if the inequality is to be satisfied, since odd powers will produce a negative real number.

$$\frac{1024}{625} \approx 1.64^-$$

$$\text{Since } 1.6^4 = 2.56^2 < 2.6^2 = 6.76 \text{ and } 1.65^4 < 2.73^2 < 2.8^2 = 7.84$$

Thus, $k = 4$ is too small, since the value of the expression lies between 6.76 and 7.84.

Suspect $k = 6$, but let's check $k = 5$ (ignoring the minus sign) just to be sure.

$$\text{Using an underestimate, } 1.6^5 = \left(\frac{16}{10} \right)^5 = \frac{2^{20}}{10^5} = \frac{2^{15}}{5^5} = \frac{2^{10} 2^5}{5^5} = \frac{1024(32)}{3125} = \frac{32768}{3125} > 10 > 8.$$

So, clearly, $k = \mathbf{6}$ produces an even larger value and is the required minimum value of k .

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2013 SOLUTION KEY**

Team Round

B) Let (t, u) denote the tens' and units' digits of the two-digit number (in base 10).

The original two-digit integer is $10t + u$.

The positive difference of the digits is either $t - u$ or $u - t$.

According to the first condition, if $t > u$, $\frac{10t+u}{t-u} = 21$ or if $t < u$, $\frac{10t+u}{u-t} = 21$.

$$t > u \Rightarrow 10t + u = 21t - 21u \Rightarrow 22u = 11t \Rightarrow t = 2u$$

$$t < u \Rightarrow 10t + u = 21u - 21t \Rightarrow 31t = 20u \text{ has no solutions for base 10 digits.}$$

According to the second condition, $tu + (t - u) = 21$.

Substituting, $2u^2 + u - 21 = (2u + 7)(u - 3) = 0 \Rightarrow u = 3$ only $\Rightarrow N = \underline{63}$ only.

C) $AS = 6 \Rightarrow$ area of $\triangle ABC$ is 48

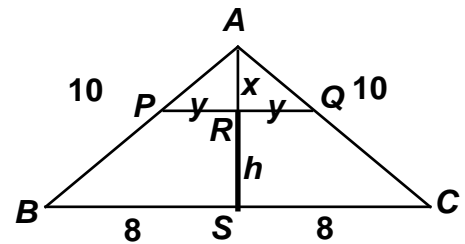
\Rightarrow the area of $\triangle APQ = 8$.

$$\overline{PQ} \parallel \overline{BC} \Rightarrow \triangle APR \sim \triangle ABS \Rightarrow \frac{x}{y} = \frac{6}{8} \Rightarrow 4x = 3y.$$

$$\frac{1}{2}x(2y) = xy = 8$$

$$12xy = (4x)(3y) = 12 \cdot 8 = 96$$

Substituting, $16x^2 = 96 \Rightarrow x = \sqrt{6} \Rightarrow h = \underline{6 - \sqrt{6}}$.



**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2013 SOLUTION KEY**

Team Round

$$\begin{aligned} \text{D) } x^{14} - x^8 - x^6 + 1 &= (x^6 - 1)(x^8 - 1) = (x^3 + 1)(x^3 - 1)(x^4 + 1)(x^4 - 1) \\ &= (x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1)(x^4 + 1)(x^2 + 1)(x + 1)(x - 1) \end{aligned}$$

Thus, the sum of the factors is $x^4 + 3x^2 + 4x + 4 \Rightarrow \underline{\mathbf{(1, 3, 4, 4)}}$.

$$\begin{aligned} \text{E) } \frac{\cot 2x \cdot \cot x + 1}{\cot x - \cot 2x} &= \frac{\frac{1 - \tan^2 x}{2 \tan x} \cdot \frac{1}{\tan x} + 1}{\frac{1}{\tan x} - \frac{1 - \tan^2 x}{2 \tan x}} \cdot \frac{2 \tan^2 x}{2 \tan^2 x} = \frac{1 - \tan^2 x + 2 \tan^2 x}{2 \tan x - \tan x(1 - \tan^2 x)} \\ &= \frac{1 + \tan^2 x}{\tan x + \tan^3 x} = \frac{\cancel{1 + \tan^2 x}}{\tan x (\cancel{1 + \tan^2 x})} = \frac{1}{\tan x} = \cot x \end{aligned}$$

Thus, we have $\cot x = \tan 300^\circ = -\tan 60^\circ = -\cot 30^\circ$.

The 30° family over the specified domain is $\{30^\circ, 150^\circ, 210^\circ, 330^\circ\}$.

The solution set consists of only values in quadrants 2 and 4, where the cotangent takes on a negative value, namely, $x = \underline{\mathbf{150^\circ, 330^\circ}}$.

F) Since the interior and exterior angles in a regular polygon with n sides are given by $\frac{180(n-2)}{n}$ and $\frac{360}{n}$ respectively, the given ratios translate to $\frac{n-2}{2} = \frac{11}{q}$ and $\frac{2}{m-2} = \frac{1}{11}$.

Thus, $q = \frac{22}{n-2}$ and $m = 24$ and the exterior angles must be 15° .

The required ratio is $\frac{180(n-2)}{15} = 12\left(1 - \frac{2}{n}\right) = 12 - \frac{24}{n}$ and

n must be a factor of 24 (≥ 3 of course).

Thus, $n = 3, 4, 6, 8, 12$ and 24 are under consideration and only $3, 4$ and 24 produce integer values of q . Therefore, $(n, q) = \underline{\mathbf{(3, 22), (4, 11) \text{ and } (24, 1)}}$.