MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 – DECEMBER 2013
ROUND 1 TRIG: RIGHT ANGLE PROBLEMS, LAWS OF SINES AND COSINES

ANSWERS

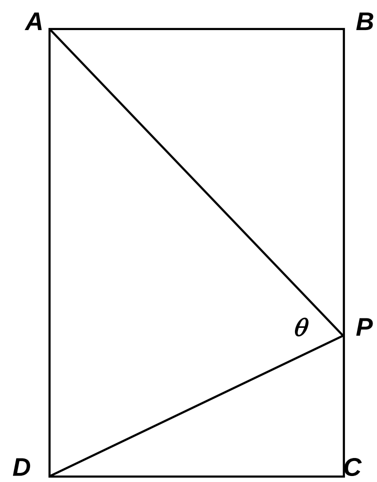
A) _______________________________

B) _______________________________

C) _______________________________

A) The sides of right ΔABC are 1, x and 7, where $1 < x < 7$. A is the larger acute angle. Compute the $\tan(\angle A)$.

B) In rectangle $ABCD$, $AB = 24$ and $BC = 42$. Point P is located on $\overline{BC}$ such that $BP : PC = 16 : 5$. Compute $\sin\theta$.



C) ΔABC has sides in the ratio of 4 : 5 : 6.
If the area of ΔABC is $375\sqrt{7}$, then compute the perimeter of ΔABC.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013
ROUND 2 ARITHMETIC/NUMBER THEORY

ANSWERS

A) ______________________________

B) 20_______

C) ______________________________

A) A and B are perfect squares. There are no perfect squares between A and B. If both A and B are 3-digit integers, what is the <u>maximum</u> value of $A - B$?

B) Today (12-5-2013) falls on a Thursday. In what year does 12-5 next fall on a Thursday?

C) Let d be the smallest odd digit that does not appear in the decimal equivalent of $\frac{1}{7}$.

Consider a list of all positive odd 4-digit integers N with distinct digits which satisfies these conditions:

- it is a multiple of 11
- it is a multiple of d
- it is not divisible by 88% of the 25 primes less than 100.

This list is sorted in order of increasing digitsum. Integers with the same digitsum are sorted in increasing order of magnitude. What is the <u>second</u> integer in the list?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013
ROUND 3 COORDINATE GEOMETRY OF LINES AND CIRCLES

ANSWERS

A) (____________ , ____________)

B) A(____ , ____) B(____ , ____)

C) ______________________________

A) The slope and the y-intercept of the line with equation $\frac{2x}{15}+\frac{y}{4}=1$ are m and b respectively. Compute the ordered pair (m, b).

B) $\overline{AB}$ is the diameter of the circle $4x^2+4y^2-12x+20y+18=0$ parallel to the x-axis. Compute the endpoints of A and B, given that A is to the left of B.

C) Line $\mathscr{L}$ with a slope of $-\frac{1}{2}$ passes through the point $P(13, –2)$. Line $\mathscr{L}$ is tangent to a circle with center $C(3, –2)$. Find the equation of this circle in the form $(x-h)^2+(y-k)^2=r^2$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013
ROUND 4 ALG 2: LOG & EXPONENTIAL FUNCTIONS

ANSWERS

A) ______________________________

B) $x =$ ___________________________

C) ______________________________

A) Given: For positive integers x and y, $x^y = y^x = 16$

Compute all possible values of $\dfrac{\log_x y + \log_y x}{x - y}$.

B) Given: $b > 0$ $(b \neq 1)$ and $x > 0$

Solve for x in terms of b. $\log_b x - \log_{b^3} x + \log_b \sqrt[5]{x} = 4$

C) Given: $f(x) = 2^x - 2^{-x}$

If $f(A) = 8$ and $f(B) = 4$ for $A > 0$ and $B > 0$, compute $2^A - 2^B$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013
ROUND 5 ALG 1: RATIO, PROPORTION OR VARIATION

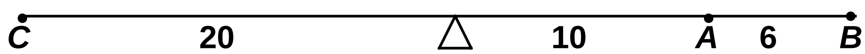
ANSWERS

A) ____________________ lbs

B) M (__________ , __________)

C) r = ____________________

A) Two children have weights of 60 lbs and 80 lbs. They sit on the right side of the seesaw at points A and B in the diagram below. A third child sits at point C and balances the seesaw. What is the maximum weight of the third child?



Note: The seesaw balances if the sum of the products of the weight and the corresponding distance from the pivot point on the right side is the equal to the product of the weight and distance from the pivot point on the left side.

B) y_1 varies directly as x, and y_2 varies inversely as x. Specifically,

$y_1 = f(x) = 3x + 8$ and . $y_2 = g(x) = \dfrac{3}{x}$

The graphs of $f(x)$ and $g(x)$ intersect in two points A and B.

Compute the coordinates of the midpoint M of $\overline{AB}$.

C) On a roundtrip training run between his home (H) and the beach (B), Rocky traveled over the same route both ways. Let R denote his rate when he ran from H to B. His average overall rate was only $\dfrac{3}{4}R$ because (due to cramps) he had to slow down returning from B to H.

In terms of R, compute r, his average rate on the return trip.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013
ROUND 6 PLANE GEOMETRY: POLYGONS (no areas)

ANSWERS

A) ________________________________

B) _______ $\le$ _______ $\le$ _______

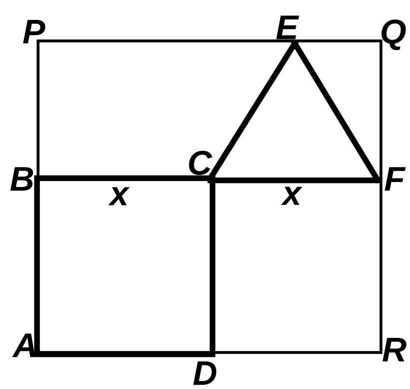
C) ________________________________

A) A regular polygon has 88 more diagonals than sides.
Compute the degree-measure of an exterior angle of this polygon.

B) Regular pentagon MINDY and regular decagon BLACKSMITH lie on opposite sides of line $\overleftrightarrow{MI}$. List the three degree-measures of the angles in ΔTIN from smallest to largest.

C) Square $ABCD$ has a side of length x.
Equilateral triangle CEF has side of length x.
Points B, C and F are collinear, as are each of these sets of three points:
P, B and A $\quad$ A, D and R $\quad$ R, F and Q $\quad$ Q, E and P

The diagonal in rectangle $PQRA$ has length d.
Compute the numerical value of the ratio $\frac{d^2}{x^2}$ as a single fraction with a rationalized denominator.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013
ROUND 7 TEAM QUESTIONS
ANSWERS

A) ____________________________ D) ____________________________

B) 1 2 3 4 5 6 7 E) ____________________________

C) ____________________________ F) ____________________________

A) The sides of a unique right triangle *ABC* are, in order of increasing magnitude, $\left[\frac{x}{8}\right]-1$, $x-1$ and x. The sides are integer lengths with no common factor (other than 1). Compute the perimeter of this triangle.
Note: $[N]$ denotes the greatest integer less than or equal to *N*.
For positive real numbers, the fractional part is truncated (dropped).

B) Definitions: $x \heartsuit y = \frac{x+y}{2}$ (arithmetic average) and $x \blacklozenge y = \frac{2xy}{x+y}$ (harmonic average).

Under which of the following condition(s) does the operation ♥ <u>distribute</u> over the operation ♦, i.e. $a \heartsuit (b \blacklozenge c) = (a \heartsuit b) \blacklozenge (a \heartsuit c)$, where *a*, *b* and *c* are non-negative integers for which <u>both</u> sides of the equality are defined. Circle your choice(s) above.

1) $a = 0$ 2) $b = 0$ 3) $c = 0$ 4) $a = b$ 5) $a = c$ 6) $b = c$ 7) none of the above

C) A line $\mathcal{L}_1$ through the point $Q(-2, 3)$ divides the circle *P* with equation $x^2 + y^2 - 8x + 10y - 23 = 0$ into two congruent regions. A line $\mathcal{L}_2$ passes through the center of the circle perpendicular to $\mathcal{L}_1$. Let *R* be the *x*-intercept of $\mathcal{L}_1$ and *S* be the *y*-intercept of $\mathcal{L}_2$ respectively. Compute the area of quadrilateral *PROS*, where *O* denotes the origin.

D) For $10 \le x \le 10000$, define the function $f(x) = x^{4-\log_{10} x}$.

Let *M* be the minimum value and *N* be the maximum value that $f(x)$ may take on. Compute $\frac{M}{N}$.

E) The area of a certain quadrilateral varies jointly as the distance *D* between two parallel sides and the sum *S* of the lengths of these parallel sides. Two such quadrilaterals are initially congruent and, therefore, have the same area. In the first quadrilateral, *D* is multiplied by a factor of $9n^2$ and *S* is unchanged. In the second quadrilateral, *S* is multiplied by a factor of $27n-20$ and *D* is unchanged. Compute all possible values of *n* for which the areas of these quadrilaterals remain equal.

F) A regular polygon has vertices $V_1, V_2, V_3, \ldots, V_n$. 17 diagonals can be drawn from each vertex. Let *P* be the point of intersection between diagonal $\overline{V_iV_{i+3}}$ and $\overline{V_{i+1}V_{i+5}}$, where $1 \le i \le 10$. Compute $m\angle V_{i+1}PV_{i+3}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013 ANSWERS

Round 1 Trig: Right Triangles, Laws of Sine and Cosine

A) $4\sqrt{3}$ B) $\frac{63}{65}$ C) 150

Round 2 Arithmetic/Elementary Number Theory

A) 61 B) 2019 C) 2013

Round 3 Coordinate Geometry of Lines and Circles

A) $\left(-\frac{8}{15},4\right)$ B) $A\left(-\frac{1}{2},-\frac{5}{2},\right)B\left(\frac{7}{2},-\frac{5}{2},\right)$ C) $(x-3)^2+(y+2)^2=20$

Round 4 Alg 2: Log and Exponential Functions

A) ± 1.25 B) $x = b^{\frac{60}{13}}$ C) $2+\sqrt{17}-\sqrt{5}$

Round 5 Alg 1: Ratio, Proportion or Variation

A) 94 B) $\left(-\frac{4}{3},4\right)$ C) $\frac{3R}{5}$ (or $\frac{3}{5}R$ or $0.6R$)
(The zero is not required.)

Round 6 Plane Geometry: Polygons (no areas)

A) 22.5 B) $36 \le 36 \le 108$ C) $\frac{23+4\sqrt{3}}{4}$
(or equivalent)
(Accept A) and B) with or without the degree symbol.)

Team Round

A) 306 D) $\frac{1}{10,000}$

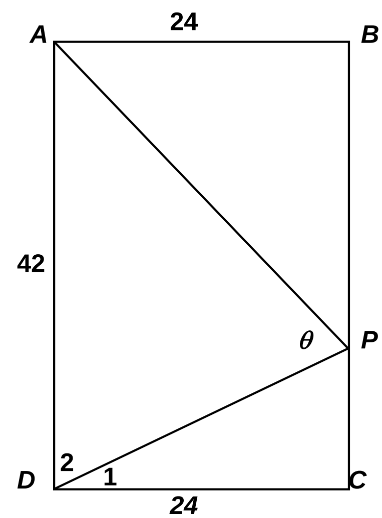
B) 1 and 6 only E) $\frac{4}{3},\frac{5}{3}$

C) $\frac{133}{8}$ (or 16.625) F) 153

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013 SOLUTION KEY

Round 1

A) Since 7 must be the length of the hypotenuse, $x = 4\sqrt{3}$.
Since A is the larger acute angle, it must be opposite the longer side.
SOHCAHTOA $\Rightarrow$ tan($\angle A$) = $\underline{\mathbf{4\sqrt{3}}}$.



B) BP : PC = 16 : 5 and BC = 42 $\Rightarrow$ BC = 32, CP = 10.
Using special Pythagorean triples of 3-4-5 and 5-12-13, we have
in ΔPCD, (10, 24, ?) = 2(5, 12, ?) $\Rightarrow$ PD = 26
in ΔABP, (24, 32, ?) = 8(3, 4, ?) $\Rightarrow$ AP = 40
In ΔPCD, cos($\angle 1$) = $\dfrac{24}{26} = \dfrac{12}{13}$

Angles 1 and 2 are complementary, so sin($\angle 2$) = cos($\angle 1$) = $\dfrac{12}{13}$.

Using the Law of Sines in ΔAPD,
$$\frac{\sin\theta}{42} = \frac{\sin\angle 2}{40} \Rightarrow \sin\theta = \frac{42}{40}\cdot\frac{12}{13} = \underline{\mathbf{\frac{63}{65}}}\ .$$

C) Let the three sides have lengths $4k$, $5k$ and $6k$. The smallest angle θ will be opposite the side of length $4k$. Using the law of Cosines, $16k^2 = 25k^2 + 36k^2 - 60k^2\cos\theta°$.

$$k \ne 0 \Rightarrow 60\cos\theta = (25+36-16) = 45 \Rightarrow \cos\theta = \frac{3}{4} \Rightarrow \sin\theta = +\frac{\sqrt{7}}{4} \text{ (since } \theta \text{ must be acute)}$$

The area of any triangle can be computed as $\frac{1}{2}ab\sin C$, where C is the included angle.

Thus, the area of ΔABC is $\frac{1}{2}(5k)(6k)\left(\frac{\sqrt{7}}{4}\right) = 375\sqrt{7} \Rightarrow 15k^2 = 4(375)$.

$\Rightarrow$ k = 10 and the perimeter is $\underline{\mathbf{150}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013 SOLUTION KEY

Round 2

A) *A* and *B* must be squares of consecutive integers. Since the gap between consecutives squares grows as the squares get larger, we take the two largest 3-digit perfect squares.
$30^2 = 900$, $31^2 = 961$, but $32^2 = 1024$, so the maximum difference is $961 – 900 =$ **61**.

B) Today is 12/5/2013.
Let DOW denote day of the week. The DOW sequence is MonTueWedThuFriSatSunMon… .
There are 365 days in a year (unless it's a leap year, in which case February 29th makes 366 days). In a 365 day year, there are $\left[\frac{365}{7}\right]=52$ weeks, plus 1 extra day.
Thus, from one year to the next, a specific date advances one day of the week, unless there is an intervening leapday!
2012 was a leap year and 2016, 2020, 2024 will be as well.
For example, 12/5/2016 falls 2 DOWs after 12/5/2015, because of the extra day 2/29/2016.
The sequence of DOWs for 12/5, starting in 2013 is Thu, Fri(2014), Sat(2015), Mon(2016), Tue (2017), Wed(2018), Thu(**2019**).

C) $\frac{1}{7}=0.\overline{142857} \Rightarrow d = 3$
Therefore, *N* is an odd multiple of 33. There are 25 primes less than 100:
2,3,5,7, 11,13,17,19, 23,29, 31,37, 41,43,47, 53,59, 61,67, 71,73,79, 83,89, 97
88% of 25 is 22 $\Rightarrow$ *N* is divisible by exactly 3 distinct primes, including 3 and 11.
The smallest digit sum is 6 if *N* can be formed using the digits 0, 1, 2 and 3.
All other sets of possible digits with a digit sum of 6 will have at least one repeated digit.
A 4-digit integer consisting of digits 0, 1, 2 and 3 will always be divisible by 3.
There are 24 arrangements of these *N*-values divisible, only 18 are 4-digit numbers and only 12 of these are odd. Divisibility by 11 narrows the field to 2: 1023 and 2013
The smallest value 1023 = $3\cdot 11\cdot 31$ and the next smallest is 2013 = $3\cdot 11\cdot 67$. Thus, *N* = **2013**.
(Without the restriction that the digits were distinct, we would have had to consider *N*-values formed from {1, 1, 2, 2}, {1, 1, 1, 3}, {1, 1, 0, 4} and {2, 2, 2, 0}. Only the first set of digits produces a multiple of 11. In this case, the second integer in the list would have been 1221.)

We could also have proceeded by brute force, examining products of the form $3\cdot 11\cdot x$, where *x* is a prime such that $x>\left[\frac{1000}{33}\right]=30$.

31: 1023 (smallest)
37: 1221 (rejected, repeated digits)
41: 1353 (rejected, digit sum = 12)
43: 1419 (rejected, digit sum = 15)
47: 1551 (rejected, digit sum = 12)
53: 1749 (rejected digit sum = 21)
59: 1947 (rejected, digit sum = 21)
61: **2013** Bingo!

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013 SOLUTION KEY

Round 3

A) Multiplying by 60, $\frac{2x}{15}+\frac{y}{4}=1 \Leftrightarrow 8x+15y=60 \Leftrightarrow y=-\frac{8}{15}x+4$

Since the equation is now in $y=mx+b$ form, the required order pair is $\underline{\left(\mathbf{-\frac{8}{15},4}\right)}$.

B) $4\left(x^2-3x+\underline{\frac{9}{4}}\right)+4\left(y^2+5y+\underline{\frac{25}{4}}\right)=-18+9+25 \Leftrightarrow \left(x-\frac{3}{2}\right)^2+\left(y+\frac{5}{2}\right)^2=4$

$\Rightarrow$ Center: $\left(\frac{3}{2},-\frac{5}{2}\right)$ and radius 2 $\Rightarrow \left(\frac{3}{2}\pm 2,-\frac{5}{2}\right) \Rightarrow \underline{\boldsymbol{A}\left(\mathbf{-\frac{1}{2},-\frac{5}{2},}\right)\ \boldsymbol{B}\left(\mathbf{\frac{7}{2},-\frac{5}{2},}\right)}$

C) The equation of line $\mathscr{L}$ is $(y+2)=-\frac{1}{2}(x-13) \Leftrightarrow x+2y=9$

A radius through the center drawn to the point of contact will have slope +2.
Its equation is $(y+2)=2(x-3) \Leftrightarrow 2x-y=8$
The point of tangency T is the intersection of these two lines.

$\begin{cases} x+2y=9 \\ 2x-y=8 \end{cases} \Rightarrow \begin{cases} x+2y=9 \\ 4x-2y=16 \end{cases} \Rightarrow 5x=25 \Rightarrow (x,y)=(5,2)$

The radius of this circle is the distance from C to T, namely $\sqrt{(2+2)^2+(5-3)^2}=\sqrt{20}$.

Therefore, the required equation is $\underline{\mathbf{(x-3)^2+(y+2)^2=20}}$.

Alternately, the point-to-line distance formula could have been used to find the radius.

The distance from $(3,-2)$ to $x+2y-9=0$ is $\frac{|3\cdot 1+(-2)\cdot 2+(-9)|}{\sqrt{1^2+2^2}}=\frac{10}{\sqrt{5}}=2\sqrt{5}$ which is the radius of the given circle, resulting in the same equation as above.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013 SOLUTION KEY

Round 4

A) Since $2^4 = 16$ and $4^2 = 16$, we have (x, y) = (2,4) or (4, 2).

Therefore, $\dfrac{\log_x y + \log_y x}{x-y} = \dfrac{\log_2 4 + \log_4 2}{\pm 2} = \dfrac{2+\frac{1}{2}}{\pm 2} = \underline{\mathbf{\pm 1.25}}$.

How do you argue that there are no other ordered pairs of positive integers???

B) $\log_b x - \log_b\left(x^{1/3}\right) + \log_b\left(x^{1/5}\right) = 4 \Rightarrow \log_b\left(\dfrac{xx^{1/5}}{x^{1/3}}\right) = 4 \Rightarrow \log_b\left(\dfrac{x^{18/15}}{x^{5/15}}\right) = \log_b\left(x^{13/15}\right) = 4$

$\Rightarrow b^4 = x^{13/15}$ Raising each side to the 15/13th power , x = $\underline{\boldsymbol{b^{\frac{60}{13}}}}$

C) If $f(x) = 2^x - 2^{-x} = k$, then

$2^x - \dfrac{1}{2^x} = k \Leftrightarrow \dfrac{2^{2x}-1}{2^x} = k \Leftrightarrow 2^{2x} - k \cdot 2^x - 1 = 0 \Leftrightarrow \left(2^x\right)^2 - k \cdot 2^x - 1 = 0$

If $N = 2^x$, then we have $N^2 - kN - 1 = 0$ or $N = \dfrac{k \pm \sqrt{k^2+4}}{2} \Rightarrow$

For $x = A$ and k = 8, we have $N = 2^A = \dfrac{8 \pm \sqrt{8^2+4}}{2} = 4 + \sqrt{17}$.

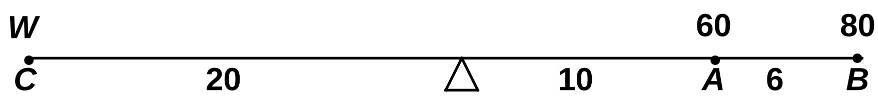
For $x = B$ and k = 4, we have $N = 2^B = \dfrac{4 \pm \sqrt{4^2+4}}{2} = 2 + \sqrt{5}$

Thus, $2^A - 2^B = 4 + \sqrt{17} - \left(2 + \sqrt{5}\right) = \underline{\mathbf{2 + \sqrt{17} - \sqrt{5}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013 SOLUTION KEY

Round 5

A) The heavier child should sit farther from the pivot point.
60(10) + 80(10 + 6) = W(20) $\Rightarrow$ 600 + 1280 = 1880 = 20W $\Rightarrow$ W = **94**.
Note: If the heavier child sits closer to the pivot point on the right side, we would have
60(10 + 6) + 80(10) = W(20) $\Rightarrow$ 960 + 800 $\Rightarrow$ W = 48 + 40 = 88, a smaller value.



B) $y_1 = 3x + 8$ and $y_2 = \frac{3}{x} \Rightarrow$

$x(3x+8) = 3 \Leftrightarrow 3x^2 + 8x - 3 = (3x-1)(x+3) = 0 \Rightarrow x = \frac{1}{3}, -3 \Rightarrow A\left(\frac{1}{3}, 9\right), B(-3,-1)$

Applying the midpoint formula, $M\left(\frac{\frac{1}{3}+(-3)}{2}, \frac{9+(-1)}{2}\right) = \underline{\left(-\frac{\mathbf{4}}{\mathbf{3}}, \mathbf{4}\right)}$.

C) His overall average rate is the harmonic average of his rates $\left(\frac{2r_1r_2}{r_1+r_2}\right)$ - NOT the arithmetic average $\left(\frac{r_1+r_2}{2}\right)$. A good topic of discussion with your teammates/coach!
Since r denotes his average return rate, we have
$\frac{2rR}{r+R} = \frac{3}{4}R = \frac{3R}{4}$ Cross multiplying, $8rR = 3Rr + 3R^2$. Dividing through by R,
$8r = 3r + 3R \Rightarrow 5r = 3R$ or $r = \underline{\frac{\mathbf{3R}}{\mathbf{5}}}$.
(Acceptable alternative forms are listed on the answer key.)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013 SOLUTION KEY

Round 6

A) A regular polygon with n sides (sometimes referred to as an n-gon) has $\frac{n(n-3)}{2}$ diagonals and the exterior angle contains $\frac{360°}{n}$. Thus, we require that

$$\frac{n(n-3)}{2} = n+88 \Leftrightarrow n^2 - 5n - 176 = 0$$

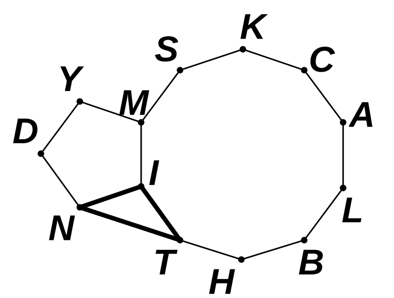
Factoring, we have $(n+11)(n-16) = 0 \Rightarrow n = 16$ and the exterior angle measure is $\frac{360}{16}$ = $\underline{\boldsymbol{\frac{45}{2}}°}$ ($\underline{\boldsymbol{22\frac{1}{2}}°}$ or **<u>22.5°</u>**).

B) The angles of a regular pentagon are each $\frac{180(5-2)}{5} = 108°$; the angles of a regular decagon are $\frac{180(10-2)}{10} = 144°$. Therefore,

$m\measuredangle TIN = 360 - (m\measuredangle MIN + m\measuredangle MIT) = 360 - (108+144) = 108$.

Since ΔTIN is isosceles, its base angles each measure $\frac{180-108}{2} = \frac{72}{2} = 36$.

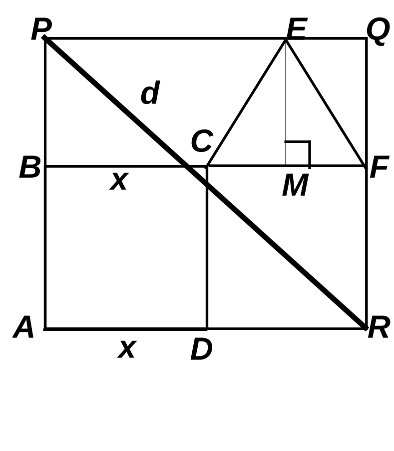
Thus, the required sequence is **<u>36, 36, 108</u>**.



C) Since $EM = \frac{x}{2}\sqrt{3}$, $AR = 2x$, we have

$$\left(x\left(1+\frac{\sqrt{3}}{2}\right)\right)^2 + (2x)^2 = d^2 \Rightarrow \frac{d^2}{x^2} = 4 + \left(1+\frac{\sqrt{3}}{2}\right)^2$$

$$= 4 + 1 + \sqrt{3} + \frac{3}{4} = \frac{23}{4} + \sqrt{3} = \underline{\boldsymbol{\frac{23+4\sqrt{3}}{4}}}$$



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013 SOLUTION KEY

Team Round

A) Your first line of attack might be to try the Pythagorean Theorem.

$$\left(\left[\frac{x}{8}\right]-1\right)^2+\left(x-1\right)^2=x^2 \Rightarrow \left(\left[\frac{x}{8}\right]-1\right)^2=2x-1 \Rightarrow \left[\frac{x}{8}\right]^2-2\left[\frac{x}{8}\right]+1$$

$$\Rightarrow 2\left(x-1\right)=\left[\frac{x}{8}\right]\left(\left[\frac{x}{8}\right]-2\right)$$

But then what??

A quicker solution would be to list the Pythagorean triples where the length of the hypotenuse is 1 more than the long leg and note a pattern.

3	4	5
5	12	13
7	24	25
9	40	41

The gap between the long legs is growing by 4 as the gap between the short legs remains constant at 2. Since $\left[\frac{5}{8}\right]-1\neq 3$, $\left[\frac{13}{8}\right]-1\neq 5$, $\left[\frac{25}{8}\right]-1\neq 7$, and $\left[\frac{41}{8}\right]-1\neq 9$, we must continue the pattern.

11	60	61	$\left[\frac{61}{8}\right]-1=7-1=6\neq 11$
13	84	85	$\left[\frac{85}{8}\right]-1=10-1=9\neq 13$
15	112	113	$\left[\frac{113}{8}\right]-1=14-1=13\neq 15$
17	144	145	$\left[\frac{145}{8}\right]-1=18-1=17$ Bingo!

Therefore, the perimeter of ΔABC is $17+144+145=\underline{\mathbf{306}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013 SOLUTION KEY

Team Round

B) Definitions: $x \heartsuit y = \dfrac{x+y}{2}$ (arithmetic average) and $x \diamondsuit y = \dfrac{2xy}{x+y}$ (harmonic average)

Method #1:

Let $(a, b, c) = (1, 0, 1)$. (Testing conditions 2 and 5)

$1 \heartsuit (0 \diamondsuit 1) = 1 \heartsuit 0 = 1/2$ and $(1 \heartsuit 0) \diamondsuit (1 \heartsuit 1) = (1/2) \diamondsuit 1 = 2/3$

Since the results are unequal, in general, conditions 2) and 5) fail.

Let $(a, b, c) = (1, 1, 0)$. (Testing conditions 3 and 4)

$1 \heartsuit (1 \diamondsuit 0) = 1 \heartsuit 0 = 1/2$ and $(1 \heartsuit 1) \diamondsuit (1 \heartsuit 0) = 1 \diamondsuit (1/2) = 2/3$

Since the results are unequal, in general, conditions 3) and 4) fail.

All my attempts to eliminate 1 and/or 6 have failed. I can *assume* **1 and 6** always results in equality.
But what if I missed ordered triples which would have eliminated one or both conditions?

Method #2: (brute force substitution)

Compute formulas for $a \heartsuit (b \diamondsuit c)$ and $(a \heartsuit b) \diamondsuit (a \heartsuit c)$ and then substitute for each of the conditions and find formulas for each expression.

$$a \heartsuit (b \diamondsuit c) = a \heartsuit \frac{2bc}{b+c} = \frac{a+\frac{2bc}{b+c}}{2} = \frac{ab+ac+2bc}{2(b+c)} \quad (\#1)$$

$$(a \heartsuit b) \diamondsuit (a \heartsuit c) = \frac{2\left(\frac{a+b}{2}\right)\left(\frac{a+c}{2}\right)}{\left(\frac{a+b}{2}\right)+\left(\frac{a+c}{2}\right)} = \frac{(a+b)(a+c)}{2a+b+c} \quad (\#2)$$

Both sides are defined provided , provided $b \neq -c$ and $a \neq -\dfrac{b+c}{2}$. Verdict

1) $a=0$	$0 \heartsuit (b \diamondsuit c) = \dfrac{2bc}{2(b+c)} = \dfrac{bc}{b+c}$	$(0 \heartsuit b) \diamondsuit (0 \heartsuit c) = \dfrac{bc}{b+c}$	ok
2) $b=0$	$a \heartsuit (0 \diamondsuit c) = \dfrac{ac}{2c} = \dfrac{a}{2}$	$(a \heartsuit 0) \diamondsuit (a \heartsuit c) = \dfrac{a(a+c)}{2a+c}$	fails
3) $c=0$	$a \heartsuit (b \diamondsuit 0) = \dfrac{ab}{2b} = \dfrac{a}{2}$ (b $\neq 0$)	$(a \heartsuit b) \diamondsuit (a \heartsuit 0) = \dfrac{(a+b)a}{2a+b}$	fails
4) $a=b$	$b \heartsuit (b \diamondsuit c) = \dfrac{b^2+3bc}{2(b+c)}$	$(b \heartsuit b) \diamondsuit (b \heartsuit c) = \dfrac{2b(b+c)}{3b+c}$	fails
5) $a=c$	$c \heartsuit (b \diamondsuit c) = \dfrac{c^2+3bc}{2(b+c)}$	$(c \heartsuit b) \diamondsuit (c \heartsuit c) = \dfrac{2c(c+b)}{b+3c}$	fails
6) $b=c$	$a \heartsuit (c \diamondsuit c) = \dfrac{2ac+2c^2}{4c}$	$(a \heartsuit c) \diamondsuit (a \heartsuit c) = \dfrac{(a+c)(a+c)}{2a+c+c}$	
	$= \dfrac{2c(a+c)}{4c} = \dfrac{a+c}{2}$ $(c \neq 0)$	$= \dfrac{(a+c)^2}{2(a+c)} = \dfrac{a+c}{2}$ $(a+c \neq 0)$	ok

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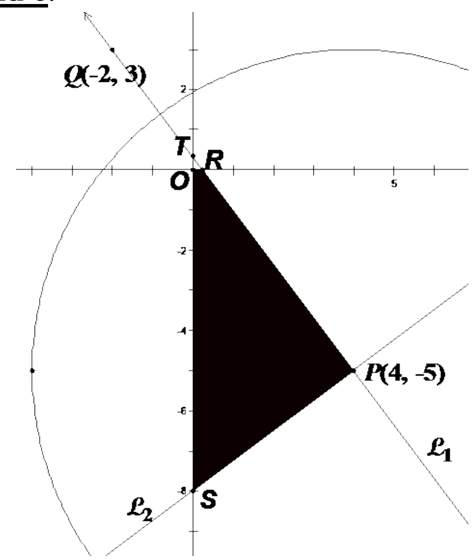
B) continued

Method #3: In general, a ♥ $(b$ ♦ $c) = (a$ ♥ $b)$ ♦ $(a$ ♥ $c) \Leftrightarrow \dfrac{ab+ac+2bc}{2(b+c)} = \dfrac{(a+b)(a+c)}{2a+b+c}$

Instead of substituting for 6 special cases, algebraically manipulate the equality.
Cross multiplying, we have equality if and only if
$(ab + ac + 2bc)(2a + b + c) = 2(b + c)(a + b)(a + c)$ or
$\cancel{2a^2b} + \cancel{\mathbf{2a^2c}} + 4abc + ab^2 + abc + \underline{2b^2c} + abc + ac^2 + \underline{\mathbf{2bc^2}} =$
$2(a^2 + ac + ab + bc)(b + c) = \cancel{2a^2b} + 2abc + 2ab^2 + \underline{2b^2c} + \cancel{\mathbf{2a^2c}} + 2ac^2 + 2abc + \underline{\mathbf{2bc^2}}$
$\Leftrightarrow 6abc + ab^2 + ac^2 = 4abc + 2ab^2 + 2ac^2$
$\Leftrightarrow 0 = -2abc + ab^2 + ac^2$
$\Leftrightarrow 0 = a(b^2 - 2bc + c^2) = 0$
$\Leftrightarrow 0 = a(b - c)^2 \Leftrightarrow a = 0$ or $b = c$
Thus, the distributive property is satisfied under conditions **<u>1 and 6</u>**.



C) $x^2 + y^2 - 8x + 10y - 23 = 0 \Leftrightarrow (x-4)^2 + (y+5)^2 = 64$

Since the line $\mathcal{L}$ must divide the circle into 2 semi-circles, it must pass through the center of the circle. Thus, $\mathcal{L}_1$ passes through $Q(-2, 3)$ and $P(4, -5)$. Its equation is

$(y-3) = \dfrac{-5-3}{4-(-2)}(x+2) \Leftrightarrow y-3 = \dfrac{-4}{3}(x+2) \Leftrightarrow 4x+3y=1$

(slope $-\dfrac{4}{3}$). $\mathcal{L}_2$ has slope $+\dfrac{3}{4}$, passes through $(4, -5)$ and has equation $3x-4y=32$. The y-intercepts are $\dfrac{1}{3}$ and -8.

The area of quadrilateral $PROS$ equals the area of ΔPTS minus the area of ΔORT, namely,

$$\frac{1}{2}\cdot\left(\frac{20}{3}\right)\cdot 5 - \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{3} = \frac{50}{3} - \frac{1}{24} = \frac{399}{24} = \mathbf{\frac{133}{8}} = \underline{\mathbf{16.625}}.$$

The following procedure works for <u>any convex polygon</u> whose vertices are known.
Start at any vertex and list the vertices in order (clockwise or counterclockwise – your choice).
Repeat the coordinates of the starting vertex. The area is given by ***half the absolute value of the sum of the downward diagonal products minus the sum of the upward diagonal products***.

$$\frac{1}{2}\begin{vmatrix} 0 & 0 \\ 1/4 & 0 \\ 4 & -5 \\ 0 & -8 \\ 0 & 0 \end{vmatrix} \Rightarrow \frac{1}{2}\left|\left(0\cdot 0 + \frac{1}{4}\cdot -5 + 4\cdot -8 + 0\cdot 0\right) - \left(0\cdot -8 + 0\cdot -5 + 4\cdot 0 + \frac{1}{4}\cdot 0\right)\right| = \frac{1}{2}\left|-\frac{5}{4} - 32\right| = \mathbf{\frac{133}{8}} = \underline{\mathbf{16.625}}$$

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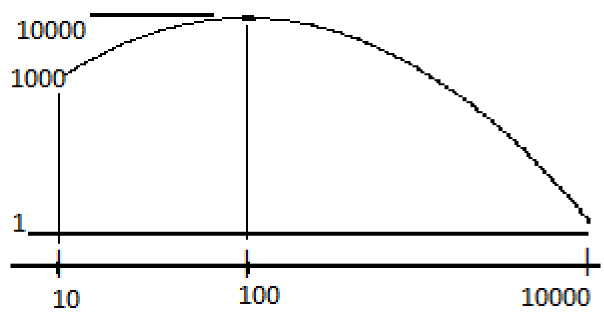
D) $f(x) = x^{4-\log_{10} x}$ By inspection, we are tempted to let x = 10, getting $M = f(10) = 10^{4-1} = 1000$, let x = 10000, getting $N = f(10000) = 4^{4-4} = 1$ and assume these are the maximum and minimum values respectively. However, this is not the case.

Taking the logarithm (base 10) of both sides, we have
$$\log_{10} f(x) = (4 - \log_{10} x)\log_{10} x$$
$$= -(\log_{10} x)^2 + 4\log_{10} x$$

This is a quadratic expression in $\log_{10} x$. For clarity, let $A = \log_{10} x$. Completing the square, we have $-A^2 + 4A = -(A^2 - 4A \underline{+4}) \underline{+4} = -(A-2)^2 + 4$ Think parabola opening down with maximum value at the vertex (2, 4). Thus, the function attains a maximum when $A = \log_{10} x = 2$ or x = 100.



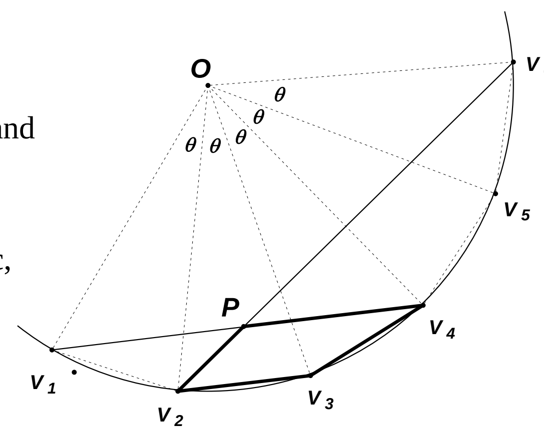
$f(100) = 100^{4-2} = 10000$ and $\dfrac{M}{N} = \underline{\dfrac{\mathbf{1}}{\mathbf{10000}}}$.

E) Joint variation implies as the product.
Let A_1 and A_2 denote the areas of the transformed quadrilaterals.
The new area A_1 is given by $k(9n^2 D)S$, while $A_2 = kD(27n - 20)S$

$$A = A' \Leftrightarrow \cancel{k}(9n^2\cancel{D})\cancel{S} = \cancel{k}\cancel{D}(27n-20)\cancel{S} \Rightarrow$$

$$9n^2 - 27n + 20 = 0 \Leftrightarrow (3n-4)(3n-5) = 0 \Rightarrow n = \underline{\mathbf{\frac{4}{3}, \frac{5}{3}}}$$

F) If there are 17 diagonals from each vertex, then $n - 3 = 17$ and the polygon has 20 sides.
Here's a sketch of the pertinent portion of the 20-gon.



Each side of the 20-gon subtends (cuts off) $\dfrac{360}{20} = 18°$ of arc, i.e. $\dfrac{1}{20}$ of the circumscribed circle. Therefore, each central angle θ measures 18°. We require $m\angle V_2PV_4$.

Consider quadrilateral $PV_2V_3V_4$. At V_2, the inscribed angle $V_6V_2V_3$ measures $\dfrac{1}{2}(3\cdot 18) = 27$.

At V_3, $m\angle V_2V_3V_4 = \dfrac{1}{2}(18\cdot 18) = 162$. At V_4, $m\angle V_1V_4V_3 = \dfrac{1}{2}(2\cdot 18) = 18$.

Therefore, $m\angle P = (360 - 27 - 162 - 18) = 360 - 207 = \underline{\mathbf{153}}°$.
Alternate solution: (Angle formed by two chords in a circle)
Let x denote an arc cut off by two successive vertices of the 20-gon. $x = 360/20 = 18$
The pair of vertical angles at P in which we are interested cut off (subtend) arcs of $2x$ and $15x$.

$$m\angle P = \frac{1}{2}(2\cdot 18 + 15\cdot 18) = \frac{1}{2}\cdot 17\cdot 18 = 17\cdot 9 = \underline{\mathbf{153}}.$$