MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2014
ROUND 1 ANALYTIC GEOMETRY: ANYTHING

ANSWERS

A) ______________________________

B) ______________________________

C) (_____ , _____) (_____ , _____)

A) The points $A(3, b)$ and $B(a, 4)$ lie on the parabola $y = x^2$.
Compute the <u>largest</u> possible value for the distance AB.

B) An ellipse is centered at the focus of the parabola $y = x^2$ and passes through the vertex and the endpoints of the focal chord of the parabola. Compute $\frac{c}{a}$, the eccentricity of this ellipse.

C) Find the coordinates of the foci for the conic defined by $3y^2 - x^2 + 24y + 14x = 49$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014
ROUND 2 ALG 1: FACTORING AND/OR EQUATIONS INVOLVING FACTORING

ANSWERS

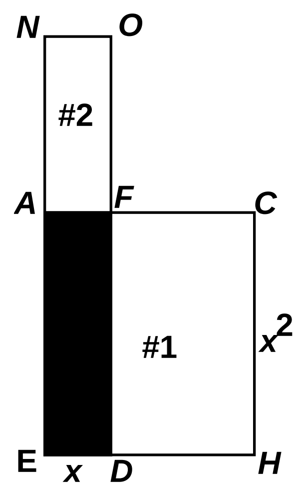
A) ________________________________

B) __________________________ units^2

C) ________________________________

A) For several positive integer values of k, the trinomial $x^2 - kx + 180$ can be factored. Determine the minimum value of k.

B) Consider the two overlapping rectangles *EACH* and *NODE*. Compute the area of the shaded region if $NE = 45$, $HE = 14$ and the area of region #1 – area of region #2 equals the area of the shaded region *DEAF*.



C) Given: $-x(2x - 3y - 4) = y^2 + 2y$
Solve for y in terms of x.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014
ROUND 3 TRIG: EQUATIONS WITH A RESAONABLE NUMBER OF SOLUTIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Compute <u>all</u> possible values of x for which $\sin\left(x^2 - x\right) = 1$ and $x^2 - x$ is the smallest positive angle measure (in degrees).

For the following problems, your answer(s) must be in radians.

B) Solve for x over $0 \le x < \dfrac{\pi}{2}$. $\sec(2x)\csc(2x) = -4$

C) Solve for x over $0 \leq x < \pi$: $3\left(\sin x - \cos x\right) + 4\left(\cos^3 x - \sin^3 x\right) = 0$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014
ROUND 4 ALG 2: QUADRATIC EQUATIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Let *P* be the positive difference between the roots of $x^2 - 13x + 30 = 0$.
Let *Q* be the positive difference between the roots of $x^2 - 13x - 30 = 0$.
Compute *P* + *Q*.

B) The sum of the roots of $2Ax^2 - Bx + C = 0$ is *AC*.
If *A*, *B* and *C* are integers and *C* > 10, compute the minimum value of *C* for which *B* is guaranteed to be a perfect square.

C) Solving a radical equation sometimes requires squaring both sides, rearranging the terms and squaring both sides again. However, doing this can introduce extraneous answers. Applying this strategy to $5 - \sqrt{x+2} = 3\sqrt{x-5}$ produces an extraneous integer solution, as well as a fractional solution that checks. Compute the fractional solution and leave your answer as a simplified ratio of integers.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014
ROUND 5 GEOMETRY: SIMILARITY OF POLYGONS

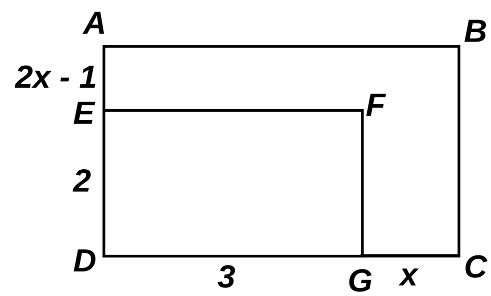
ANSWERS

A) ______________________________

B) ______________________________

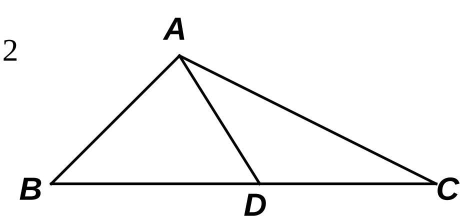
C) ______________________________

A) Compute the value of *x* for which the two rectangles are similar, given that $AE < ED$.



B) We know that all squares are similar. Consider two squares whose sides are $1+x$ and $2-x$. Compute the two values of *x* for which the ratio of the areas of these squares is 9 : 4.

C) Given: $m\measuredangle DAC = m\measuredangle DCA$, $\Delta BAC \sim \Delta BDA$, $AB = 1$, $AC = 2$
Compute *AD*.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014
ROUND 6 ALG 1: ANYTHING

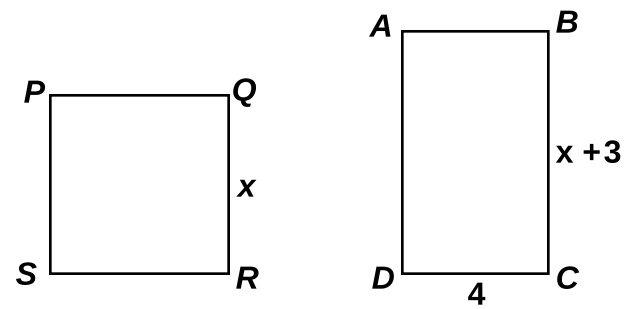
ANSWERS

A) ______________________ units

B) ______________________

C) (__________ , __________)

A) Square *PQRS* has the same area as rectangle *ABCD*. Compute the perimeter of rectangle *ABCD*.



B) You run the 100 yard dash in N seconds, where N is an integer. Suppose you could keep up this pace for one mile. Compute the largest value of N which allows you to break the 4 minute mile and join the ranks of Roger Bannister and other world-class runners who have accomplished this incredible feat.

Recall: 1 mile equals 5280 feet

C) For a real constant c, the straight line defined by $\frac{x}{3}+\frac{y}{4}=c$ has intercepts that are 7.2 units apart. Compute the coordinates of the intercept farthest from the origin.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014
ROUND 7 TEAM QUESTIONS

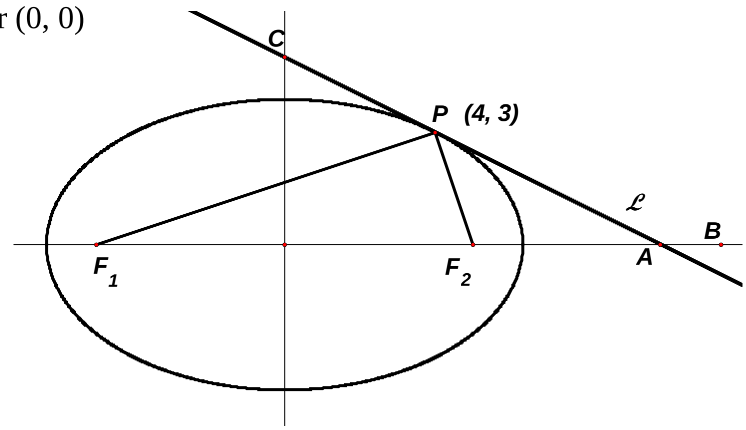
ANSWERS

A) ____________________________ D) ____________________________

B) ____________________________ E) ____________________________

C) ____________________________ F) ____________________________

A) In the figure at the right, the ellipse has center (0, 0) and vertices on the x-axis.
Line $\mathcal{L}$ is tangent to the ellipse at $P(4, 3)$ and $m\angle F_1PF_2 = 90°$.
Determine the equation of $\mathcal{L}$. Express it in $Ax + By = C$ form, where A, B and C are integers and $A > 0$.



B) Find all real values of x for which $\dfrac{4x-1}{4} = \dfrac{1}{\sqrt{4x+1}}$.

C) Solve for x over the interval $-2\pi \le x < 0$: $\cos^4 x + 6\cos^2 x + 7\cos x + \cos 3x = -\dfrac{15}{16}$

D) Find all positive integer values of k for which $x^2 + kx + k + 11$ can be expressed as the product of two linear binomials with integer coefficients.

E) In ΔPQR, $\overline{ST} \parallel \overline{PQ}$, where S lies on $\overline{PR}$ and T lies on $\overline{RQ}$.
PS = 3 units and $RS = k$ units and the area of ΔRST is 100 square units. Compute all possible positive integer values of k for which the area of trapezoid $STQP$ is also an integer.

F) Let A be a positive integer. For a specific value of A, the complex fraction $\dfrac{10(x+4)}{\dfrac{5}{x-2} - \dfrac{2}{x-A}}$ is undefined for exactly three different integer values of x.
Find the minimum sum of these three values of x.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014 ANSWERS

Round 1 Analytic Geometry: Anything

A) $5\sqrt{2}$ B) $\frac{\sqrt{3}}{2}$ C) (7, –12) and (7, 4)

Round 2 Alg: Factoring

A) 27 B) 125 C) $y = 2x$ or $y = x - 2$

Round 3 Trig: Equations

A) –9, 10 B) $\frac{7\pi}{24}, \frac{11\pi}{24}$ C) $\frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}$

Round 4 Alg 2: Quadratic Equations

A) 24 B) 18 C) $\frac{89}{16}$

Round 5 Geometry: Similarity

A) $\frac{3}{4}$ B) $\frac{1}{5}, \frac{4}{5}$ C) $\frac{2\sqrt{3}}{3}$

Round 6 Alg 1: Anything

A) 26 B) 13 C) (0, 5.76)

Team Round

A) $x + 2y = 10$

B) $\frac{3}{4}$ only

C) $-\frac{2\pi}{3}, -\frac{4\pi}{3}$

D) 9, 10 and 15

E) 1, 2, 3, 5, 6, 10, 15, 30

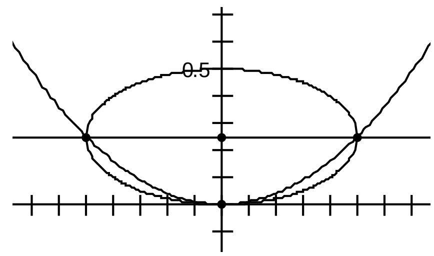
F) 14

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014 SOLUTION KEY

Round 1

A) Substituting, $b = 3^2 \Rightarrow A(3, 9)$; $4 = a^2 \Rightarrow B_1(+2, 4)$ or $B_2(-2, 4)$
The longer distance is $AB_2 \Rightarrow \sqrt{(3-^-2)^2+(9-4)^2} = \sqrt{2(5)^2} = \underline{\mathbf{5\sqrt{2}}}$

B) $y = 4px^2 = 1x^2 \Rightarrow p = \dfrac{1}{4}$



The focus of the parabola is at $\left(0, \dfrac{1}{4}\right)$.
Its vertex is at (0, 0) and the endpoints of its focal chord are located $\dfrac{1}{2}$ unit to the right and left. Thus, for the ellipse,
$a = \dfrac{1}{2}$ (the semi-major axis), $b = \dfrac{1}{4}$ (the semi-minor axis) and
for all ellipses, $a^2 = b^2 + c^2 \Rightarrow c^2 = \dfrac{3}{16} \Rightarrow c = \dfrac{\sqrt{3}}{4}$
The eccentricity (defined as $\dfrac{c}{a}$) is $\underline{\mathbf{\dfrac{\sqrt{3}}{2}}}$.

C) Completing the square,
$3y^2 - x^2 + 24y + 14x = 49 \Leftrightarrow 3\left(y^2+8y+16\right) - \left(x^2 - 14x + 49\right) = 49 + 48 - 49 \Leftrightarrow$
$3(y+4)^2 - (x-7)^2 = 48 \Leftrightarrow$
$\dfrac{(y+4)^2}{16} - \dfrac{(x-7)^2}{48} = 1$
Thus, the conic is a hyperbola with center at (7, –4) and a vertical major axis.
Since $a^2 = 16$, $b^2 = 48$ and $c^2 = a^2 + b^2$ for the hyperbola, $c = 8$ and
the coordinates of the foci are $\left(7, -4 \pm 8\right) \Rightarrow \underline{\mathbf{(7,4),\ (7,-12)}}$.

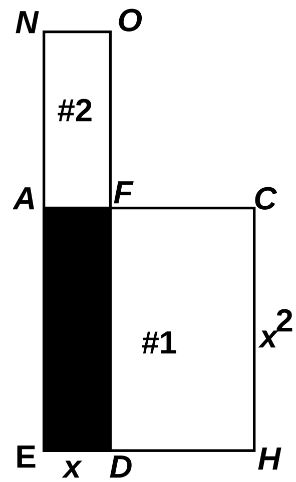
<u>FYI</u>:
At the end of this solution key, you will find additional comments about eccentricity and the conic sections - circle, ellipse, parabola and hyperbola.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014 SOLUTION KEY

Round 2

A) One possible factorization is $(x-1)(x-180)=x^2-181x+180$.
Notice that $k=181$ is the sum of the factors of 180.
Rather than trying all the factor pairs of 180, we notice that to minimize the sum of the factors we want a pair of factors as close together as possible, that is, whose difference is as small as possible. The product $12\cdot 15$ fills the bill and the minimum value of k is **27**.

B) The area of region #1 is $(14-x)x^2$.
The area of region #2 is $x\left(45-x^2\right)$.
The area of the shaded region is x^3.
Therefore, we require that $(14-x)x^2-x\left(45-x^2\right)=x^3$
$\Leftrightarrow 14x^2-x^3-45x+x^3=x^3$
$\Leftrightarrow 14x^2-45x=x^3$
Since x is nonzero, this is equivalent to $14x-45=x^2$
$x^2-14x+45=\cancel{(x-9)}(x-5)=0$ ($x\neq 9$, since CH can not be 81.)
$x=5\Rightarrow$ the area of $DEAF$ is **125**.



C) $-x(2x-3y-4)=y^2+2y\Leftrightarrow 0=\left(2x^2-3xy+y^2\right)+\left(-4x+2y\right)$
Factoring each grouping, we have $(2x-y)(x-y)-2(2x-y)=0$
Factoring out the common factor, $(2x-y)(x-y-2)=0$
Thus, $\boldsymbol{y=2x}$ or $\boldsymbol{y=x-2}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014 SOLUTION KEY

Round 3

A) The smallest positive angle is 90°. $x^2 - x = 90 \Leftrightarrow x^2 - x - 90 = (x-10)(x+9) = 0$.
Thus, $x = \underline{\mathbf{10, -9}}$.

B) $\sec(2x)\csc(2x) = -4 \Leftrightarrow \dfrac{1}{2\cos 2x \sin 2x} = -2 \Rightarrow \dfrac{1}{\sin(4x)} = -2 \Rightarrow \sin(4x) = -\dfrac{1}{2}$

$\Rightarrow 4x = \dfrac{7\pi}{6} + 2n\pi$ or $\dfrac{11\pi}{6} + 2n\pi$

$\Rightarrow x = \dfrac{(12n+7)\pi}{24}$ or $\dfrac{(12n+11)\pi}{24} \Rightarrow x = \underline{\boldsymbol{\dfrac{7\pi}{24}}}, \underline{\boldsymbol{\dfrac{11\pi}{24}}}$

C) Factoring the difference of cubes,

$3(\sin x - \cos x) + 4(\cos^3 x - \sin^3 x) = 0 \Rightarrow$

$-3(\cos x - \sin x) + 4(\cos x - \sin x)(\cos^2 x + \cos x \sin x + \sin^2 x) = 0$

$\Rightarrow (\cos x - \sin x)(-3 + 4(1 + \cos x \sin x)) = 0$

$\Rightarrow (\cos x - \sin x)(1 + 4\cos x \sin x) = 0$

$\Rightarrow (\cos x - \sin x)(1 + 2\sin 2x) = 0$

$\Rightarrow \tan x = 1$ or $\sin 2x = -\dfrac{1}{2} \Rightarrow x = \underline{\boldsymbol{\dfrac{\pi}{4}}}$ $x = 2x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6} \Rightarrow x = \underline{\boldsymbol{\dfrac{7\pi}{12}, \dfrac{11\pi}{12}}}$

If you know your identities really well, the solution is much shorter.
Multiply out and regroup as follows:

$$(4\cos^3 x - 3\cos x) + (3\sin x - 4\sin^3 x) = 0$$

These are expansions for cos(3x) and sin(3x) respectively.

$\Rightarrow \cos(3x) = -\sin(3x) \Rightarrow \tan(3x) = -1 \Rightarrow 3x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}, \dfrac{11\pi}{4}, \ldots \Rightarrow x = \underline{\boldsymbol{\dfrac{\pi}{4}, \dfrac{7\pi}{12}, \dfrac{11\pi}{12}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014 SOLUTION KEY

Round 4

A) $x^2 - 13x + 30 = (x-10)(x-3) = 0 \Rightarrow P = 10 - 3 = 7$
$x^2 - 13x - 30 = (x-15)(x+2) = 0 \Rightarrow P = 15 - (-2) = 17$
Thus, $P + Q =$ **24**.

B) The sum of the roots of $2Ax^2 - Bx + C = 0$ is $\dfrac{B}{2A}$. Therefore, we have

$\dfrac{B}{2A} = AC$ or $B = 2A^2C$

To guarantee that B is a perfect square, C must be twice a perfect square.
Thus, the candidates are: $2\left(1^2,\ 2^2,\ 3^2, \ldots.\right)$ = 2, 8, 18, … .
Since we are given that $C > 10$, we have $C =$ **18**.

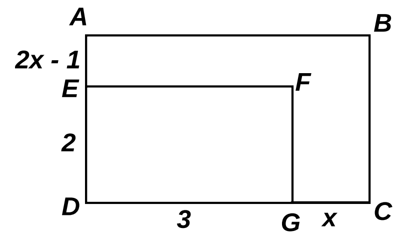
C) Assuming you didn't notice that for $x = 14$, we would have 5 – 4 = 3(3) which fails only because of the minus sign on the left side. How else could we determine that $x = 14$ is the extraneous root? Squaring both sides,

$$25 - 10\sqrt{x+2} + x + 2 = 9(x-5)$$
$$72 - 8x = 10\sqrt{x+2}$$
$$36 - 4x = 4(9-x) = 5\sqrt{x+2}$$
$$16\left(9-x\right)^2 = 25(x+2)$$
$$16x^2 - 313x + 1246 = 0$$

This looks dreadful, but we were given that there was an extraneous <u>integer</u> solution.
This gives us a partial factorization as $(16x - \boxed{A})(x - \boxed{B}) = 0$, where A and B are integers.
$A + 16B = 313$
$AB = 1246 = 2 \cdot 7 \cdot 89$
Forgetting the numerical values for a moment and considering only the parity of A and B (that is, even or odd), the first equation says A and B cannot both be even, the second equation says that A and B cannot both be odd. Therefore, they have mixed parity and in fact only A odd and B even can satisfy the first equation.
The only even factors of 1246 are 2, 14 and 178.
For $(16x - \boxed{89})(x - \boxed{14})$, we only need check the coefficient of the middle term.
$16 \cdot 14 + 89 = 224 + 89 = 313$ and we have the correct factorization.
The fractional root is $\underline{\mathbf{\dfrac{89}{16}}}$.
Check: $5 - \sqrt{\dfrac{89}{16} + 2} = 3\sqrt{\dfrac{89}{16} - 5} \Leftrightarrow 5 - \sqrt{\dfrac{121}{16}} = 3\sqrt{\dfrac{9}{16}} \Leftrightarrow 5 - \dfrac{11}{4} = 3 \cdot \dfrac{3}{4} = \dfrac{9}{4}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014 SOLUTION KEY

Round 5



A) $ABCD \sim EFGD \Rightarrow \frac{2}{3} = \frac{2x+1}{x+3} \Rightarrow 2x+6 = 6x+3 \Rightarrow x = \underline{\mathbf{\frac{3}{4}}}$

Check: $2\left(\frac{3}{4}\right)-1 = \frac{1}{2} < 2$

$ABCD \sim FGDE \Rightarrow \frac{2}{3} = \frac{x+3}{2x+1} \Rightarrow 4x+2 = 3x+9 \Rightarrow x = 7$ (extraneous) Check: $2(7)-1 = 13 \not< 2$

B) $\frac{(x+1)^2}{(2-x)^2} = \frac{9}{4} \Rightarrow 4x^2+8x+4 = 9x^2-36x+36 \Rightarrow 5x^2-44x+32 = 0$

Factoring, $(5x-4)\cancel{(x-8)} = 0$

$x = 8$ is rejected since the side of the second square would be negative.

Thus, $x = \underline{\mathbf{\frac{4}{5}}}$.

$\frac{(x+1)^2}{(2-x)^2} = \frac{4}{9} \Rightarrow 9x^2+18x+9 = 16-16x+4x^2 \Rightarrow 5x^2+34x-7 = 0$

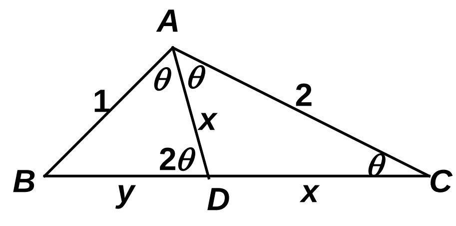
Factoring, $(5x-1)\cancel{(x+7)} = 0$

$x = -7$ is rejected since the side of the first square would be negative.

Thus, $x = \underline{\mathbf{\frac{1}{5}}}$.

C) Let $DC = x$, $BD = y$ and $m\measuredangle DAC = m\measuredangle DCA = \theta$.
As an exterior angle of ΔADC, $m\measuredangle BDA = 2\theta$
$m\measuredangle DAC = m\measuredangle DCA \Rightarrow DA = DC$
$\Delta BAC \sim \Delta BDA \Rightarrow m\measuredangle BCA = m\measuredangle BAD = \theta$

Thus, $\overrightarrow{AD}$ is an angle bisector and $\frac{y}{1} = \frac{x}{2} \Rightarrow x = 2y$.



$\Delta BAC \sim \Delta BDA \Rightarrow \frac{BC}{BA} = \frac{AC}{DA} \Leftrightarrow \frac{x+y}{1} = \frac{2}{x}$

Cross multiplying, $x^2+xy = 2 \Leftrightarrow 4y^2+2y^2 = 2 \Leftrightarrow y^2 = \frac{1}{3} \Rightarrow y = \frac{\sqrt{3}}{3} \Rightarrow AD = x = \underline{\mathbf{\frac{2\sqrt{3}}{3}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014 SOLUTION KEY

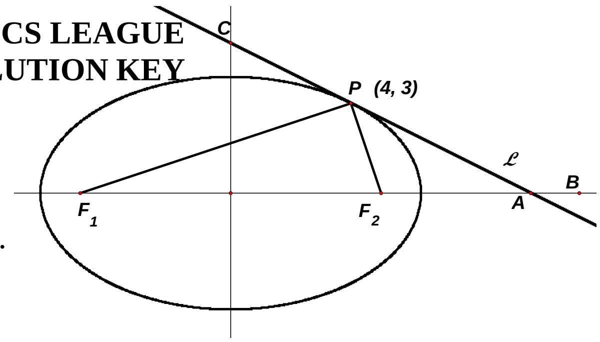
Round 6

A) $x^2 = 4(x+3) \Leftrightarrow x^2 - 4x - 12 = (x-6)(x+2) = 0 \Rightarrow x = 6$
Therefore, *ABCD* is 4 by 9 has a perimeter of **26**.

B) A mile is equivalent to $\frac{5280}{300} = \frac{528}{30} = \frac{176}{10} = 17.6$ '100 yard dashes'.
I must finish the mile in 4 minutes (240 seconds) or less. Therefore,
$17.6N < 240 \Rightarrow N < \frac{240}{17.6} = \frac{2400}{176} = \frac{600}{44} = \frac{150}{11} = 13^+$
Thus, N must be **13**.
Check: 13(17.6) = 228.8 < 240 seconds and 14(17.6) = 246.4 > 240 seconds

C) Dividing through by c, the equation is in intercept-intercept form $\frac{x}{3c} + \frac{y}{4c} = 1$.
Let X and Y denote the points on the axes corresponding to the x-intercept and the y-intercept. Let O denote the origin.
The intercepts are at $(3c, 0)$ and $(0, 4c)$ and the sides of right triangle XOY form a 3-4-5 right triangle. Thus, $5c = 7.2 \Rightarrow c = 1.44$ and Y is farthest from the origin $\Rightarrow$ **(0, 5.76)**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014 SOLUTION KEY



Team Round

A) Solution #1: (Angle of incidence equals angle of reflection.)
Let $F_1 = (-c, 0)$ and $F_2 = (c, 0)$. Let the slope of $\mathscr{L}$ be m_2.

Then: $\overline{PF_1} \perp \overline{PF_2} \Rightarrow \dfrac{3-0}{4+c} \cdot \dfrac{3-0}{4-c} = -1 \Rightarrow c^2 - 16 = 9 \Rightarrow c = 5,\ F_1(-5,0),\ F_2(5,0)$

Since $m\angle APF_2 = m\angle CPF_1$, these angles both measure 45°. The slope of $\overline{F_2P} = \dfrac{0-3}{5-4} = -3$.

The angle θ between two lines can be determined on the basis of the slopes of the two lines, namely $\left(\tan\theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}\right)$. Thus, we have $1 = \dfrac{m_2 + 3}{1 - 3m_2} \Rightarrow 1 - 3m_2 = m_2 + 3 \Rightarrow m_2 = -\dfrac{1}{2}$ and the equation of $\mathscr{L}$ is $y - 3 = -\dfrac{1}{2}(x-4) \Rightarrow$ <u>$\boldsymbol{x + 2y = 10}$</u>.

Solution #2:
As an exterior angle of ΔPF_2A, $m\angle APF_2 + m\angle PF_2A = m\angle PAB$.
The slope $\boldsymbol{m}$ of any non-vertical line equals the tangent of its angle of inclination.
Therefore,

$$\tan(\angle PAB) = \tan(\angle APF_2 + \angle PF_2A) = \frac{\tan\angle APF_2 + \tan\angle PF_2A}{1 - \tan\angle APF_2 \cdot \tan\angle PF_2A} = \frac{1 + (-3)}{1 - (1)(-3)} = -\frac{1}{2}$$

and the same result follows.

Solution #3: (**In any <u>non-right</u> triangle *ABC*,** $\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$ **.**)
Applying the theorem, in ΔAPF_2, $\tan F_2 = -3, \tan P = \tan 45° = 1$.

If $\tan A = x$, we have $x + (-3) + 1 = -3x \Rightarrow x = \dfrac{1}{2}$.

As supplementary angles, $\tan\angle PAF_2 = \dfrac{1}{2} \Rightarrow \tan\angle PAB = m_2 = -\dfrac{1}{2}$ and the same result follows.

Solution #4:
(**The tangent line to ellipse** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ **(or** $b^2x^2 + a^2y^2 = a^2b^2$ **) at *P*(*m*, *n*) is** $b^2mx + a^2ny = a^2b^2$**.**)
To get the equation of the ellipse with vertices V_1 and V_2, first note that $2a = V_1V_2 = PF_1 + PF_2$.
$PF_1^2 = 9 + 81 = 90, PF_2^2 = 9 + 1 = 10$, so $2a = 3\sqrt{10} + \sqrt{10} \Rightarrow a = 2\sqrt{10}$.

Substituting, $b^2 = a^2 - c^2 = \left(2\sqrt{10}\right)^2 - 5^2 = 15 \Rightarrow b = \sqrt{15}$

Thus, the equation of the ellipse is $15x^2 + 40y^2 = 600$, or reducing $3x^2 + 8y^2 = 120$
The equation of $\mathscr{L}$ is $3(4)x + 8(3)y = 120$ and the result follows.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014 SOLUTION KEY

Team Round

A) - **continued**

Solution #5 (Using the calculus)

The equation of the ellipse has the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $b^2x^2+a^2y^2=a^2b^2$.

Taking the implicit derivative, we can find the slope of a tangent line at any point.

$2b^2x+2a^2yy'=0 \Rightarrow y'=\frac{-b^2x}{a^2y}$. Using the strategy in solution #4 to find *a* and *b*, we have the slope of the tangent at $P(4, 3)$ is $\frac{-15(4)}{40(3)}=-\frac{1}{2}$ and the same result follows.

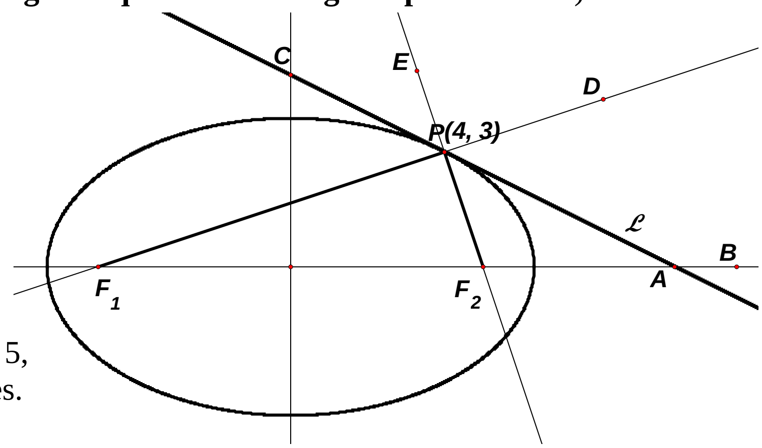
Solution #6 (**The equations of the angle bisector of the vertical angles formed by perpendicular lines can be found by adding and subtracting the equations of the given pair of lines.**)

Draw lines $\overleftrightarrow{PF_2}$ and $\overleftrightarrow{PF_1}$.
We have two pairs of vertical angles.
Line $\mathcal{L}$ bisects $\angle F_2PD$.

The equation of $\overleftrightarrow{PF_2}$ is $3x+y=15$.

The equation of $\overleftrightarrow{PF_1}$ is $x-3y=-5$.

<u>Subtracting</u> the equations,
we have $2x+4y=20$ or <u>$\mathbf{x + 2y = 10}$</u>.
Note: <u>Adding</u> the equations, we get $2x - y = 5$, the bisector of the other pair of vertical angles.



Here is the proof of this assertion on which this solution is based.

Assume the equations are $Ax+By+C=0$ and $Bx-Ay+D=0$.

Since the respective slopes are negative reciprocals, namely $-\frac{A}{B}$ and $\frac{B}{A}$, the lines are parallel. If *A* or *B* is 0, then the lines are vertical and horizontal and still perpendicular.
From geometry, we know that points on the angle bisector of an angle are equidistant from the sides of the angle. Let (x, y) be an arbitrary point of the angle bisector. Using the point-to-line distance formula, we have $r=\frac{|Ax+By+C|}{\sqrt{A^2+B^2}}$ and $r=\frac{|Bx-Ay+D|}{\sqrt{B^2+(-A)^2}}$.

Since the denominators are equal, we can ignore them. Equating the numerators,

$|Ax+By+C|=|Bx-Ay+D| \Leftrightarrow Ax+By+C=\pm(Bx-Ay+D)$

$\Leftrightarrow \begin{cases}(A-B)x+(B+A)y+(C-D)=0\\(A+B)x+(B+A)y+(C+D)=0\end{cases}$, the required result. Notice that there is no requirement that *A*, *B*, *C* and *D* be integers. So working with equations in slope intercept form, namely $y=mx+b_1$ and $y=-\frac{x}{m}+b_2$, the same results would follow.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014 SOLUTION KEY

Team Round

B) Squaring both sides and cross multiplying,

$16=(4x-1)^2(4x+1)=(4x-1)\big((4x-1)(4x+1)\big)=(4x-1)\big(16x^2-1\big)=64x^3-16x^2-4x+1$

Thus, $64x^3-16x^2-4x-15=0$

Using synthetic substitution, $\begin{array}{c|cccc} & 64 & -16 & -4 & -15 \\ & & 48 & 24 & 15 \\ \hline \frac{3}{4} & 64 & 32 & 20 & 0 \end{array}$, we have $(4x-3)(4)\big(4x^2+8x+5\big)$

and the quadratic factor has no additional real roots $\big(8^2-4\cdot 20<0\big)$. The only real root is $\underline{\boldsymbol{\frac{3}{4}}}$.

C) As an identity or considering $\cos(3x)$ as $\cos(x+2x)$ and expanding using double angle identities, we have $\cos(3x)=4\cos^3 x-3\cos x$. Then:

$$\cos^4 x+6\cos^2 x+7\cos x+\cos 3x=-\frac{15}{16}\Leftrightarrow \cos^4 x+6\cos^2 x+7\cos x+\big(4\cos^3 x-3\cos x\big)+1=\frac{1}{16}$$

Rearranging terms, $\cos^4 x+4\cos^3 x+6\cos^2 x+4\cos x+1=\frac{1}{16}$
and recognizing that the coefficients on the left side are terms in Pascal's triangle, we realize this is equivalent to
$\big(\cos x+1\big)^4=\frac{1}{16}$.

		1		
	1		1	
	1	2	1	
	1	3	3	1
1	4	6	4	1

$\Rightarrow \cos x+1=\pm\frac{1}{2}\Rightarrow \cos x=-\frac{1}{2},\cancel{-\frac{3}{2}}\Rightarrow x=\pm\frac{2\pi}{3}+2n\pi$ (reference value $\frac{\pi}{3}$ - quadrants 2 and 3)

$n=0\Rightarrow x=\underline{\boldsymbol{-\frac{2\pi}{3}}}$ $\quad n=-1\Rightarrow x=\underline{\boldsymbol{-\frac{4\pi}{3}}}$.

D) $x=\frac{-k\pm\sqrt{k^2-4k-44}}{2}=\frac{-k\pm\sqrt{(k-2)^2-48}}{2}\Rightarrow (k-2)^2-48$ must be a perfect square p.

Thus, $\big(k-2\big)^2-48=p^2\Rightarrow \big(k-2\big)^2-p^2=(k-2+p)(k-2-p)=48$

Since these two factors must be integers and have the same parity, we ignore (1)(48) and (3)(16) and restrict our attention to (2)(24), (4)(12) and (6)(8).

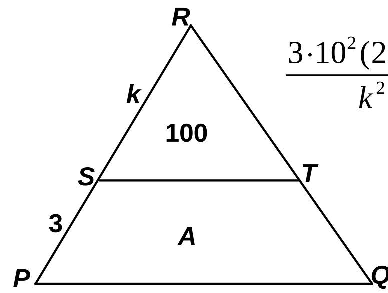
(2)(24): $(k+p)=26$ and $k-p=4\Rightarrow k=$ **<u>15</u>**
(4)(12): $(k+p)=14$ and $k-p=6\Rightarrow k=$ **<u>10</u>**
(6)(8): $(k+p)=10$ and $k-p=8\Rightarrow k=$ **<u>9</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014 SOLUTION KEY

Team Round

E) Let A denote the area of trapezoid $STQP$.

$\Delta RST \sim \Delta RPQ \Rightarrow \dfrac{area(\Delta RST)}{area(\Delta RPQ)} = \dfrac{k^2}{(k+3)^2} = \dfrac{100}{A+100}$



$\dfrac{3\cdot 10^2(2k+3)}{k^2} = \dfrac{2\cdot 2\cdot 3\cdot}{}$

Cross multiplying, $k^2A = 100(6k+9) \Rightarrow A =$

Case 1: $\dfrac{2k+3}{k}$ and $\dfrac{2^2\cdot 3\cdot 5^2}{k}$ are integers.

Since $\dfrac{2k+3}{k} = 2 + \dfrac{3}{k}$, we must consider only $k=1$ and $k=3$

k = **1** $\Rightarrow$ $A = 2^2\cdot 3\cdot 5^3 = 1500$

k = **3** $\Rightarrow$ $A = \dfrac{2^2\cdot 3\cdot 5^2\cdot 9}{3^2} = 300$

k is only a factor of $2k+3$ for k = 1 or 3.

Case 2: $\left(2^2\cdot 3\cdot 5^2\right)/k^2$ is an integer

k = **2** $\Rightarrow A = 3(5^2)(7) = 525$

k = **5** $\Rightarrow A = 2^2(5)(13) = 156$

Case 3: Combinations

k = **6** $\Rightarrow A = 2^2(5) = 125$

k = **10** $\Rightarrow A = 3(23) = 69$

k = **15** $\Rightarrow A = 4(11) = 44$

k = **30** $\Rightarrow A = 21$

Larger values of k will produce smaller values of A and since we require that $A+100$ be a perfect square, the next perfect square less than 21+100 = 121 = 11^2 is 10^2 which forces A = 0. Thus, there are no additional solutions to be found!

F) The fraction is undefined for x = 2 and $x = A$ and exactly one other integer value of x.

Simplifying, $\dfrac{10(x+4)}{\dfrac{5}{x-2} - \dfrac{2}{x-A}} = \dfrac{10(x+4)}{\dfrac{5(x-A)-2(x-2)}{(x-2)(x-A)}} = \dfrac{10(x+4)(x-2)(x-A)}{3x-5A+4}$.

Thus, the third troublesome value occurs when $3x-5A+4=0 \Rightarrow x = \dfrac{5A-4}{3}$

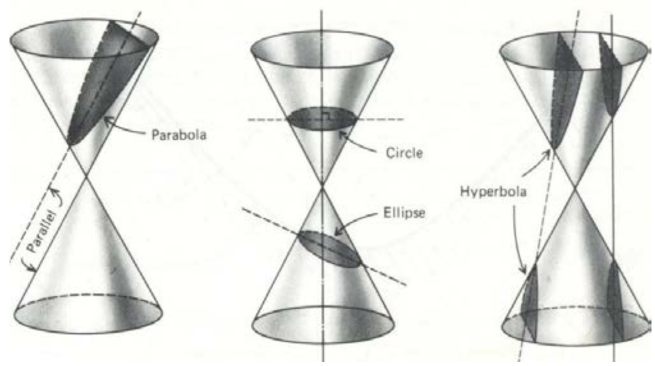
To minimize the sum, we want the smallest possible value of A. A = 2 is rejected, since this produces x = 2 which we already have. $(5A-4)$ must be a multiple of 3 and consecutive multiples of 3 differ by 3, so we increase the value of A by 3. $A = 5 \Rightarrow x = 7$.
Therefore, the minimum sum is 2 + 5 + 7 = **14**.

Alternative (brute force): Fraction is undefined for x = 2, $x = A$ and $x = \dfrac{5A-4}{3}$

$A=1 \Rightarrow x = 1,\ 2,\ 1/3$ $\quad A=2 \Rightarrow x=2$ only $\quad A=3 \Rightarrow x = 2,\ 3,\ 11/3$ $\quad A=4 \Rightarrow x = 2,\ 4,\ 16/3$

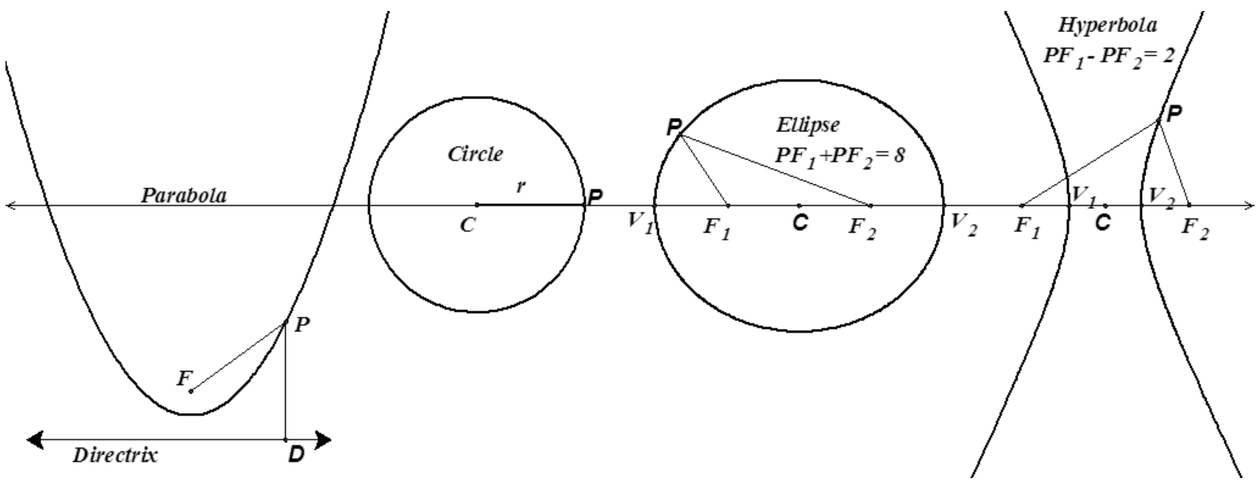
$A=5 \Rightarrow x = 2,\ 5,\ 7 \Rightarrow \min = \mathbf{14}$ In general, if $A = 3k+2$, $\dfrac{5A-4}{3}$ will be an integer and the x-sum will be $2+(3k+2)+(5k+2) = 8k+6$ for integer $k \geq 1$. $k=1$ produces the minimum.

Conic Sections and Eccentricity



The parabola, circle, ellipse and hyperbola are called conic sections because they can all be formed as cross sections of a cone (or pair of cones) as indicated in the diagrams above.

Each is understood to be a set of points in a plane satisfying a certain condition.
The conditions are usually stated as follows:
Parabola - equidistant from a fixed point (the focus) and a fixed line (the directrix)
Circle - at a fixed distance (radius) from a fixed point (center)
Ellipse - the sum of the distances from two fixed points (foci) is constant
Hyperbola – the difference of the distances from two fixed points (foci) is constant.



<u>Eccentricity</u> provides us with a different perspective on these curves and unifies the definitions.
Let F be a fixed point and $\mathscr{L}$ a fixed line. Let $d(P,\mathscr{L})$ denote the distance from point P to line $\mathscr{L}$, measured along a perpendicular drawn from P to $\mathscr{L}$.
Consider the points P in the plane for which $PF = e \cdot d(P,\mathscr{L})$.

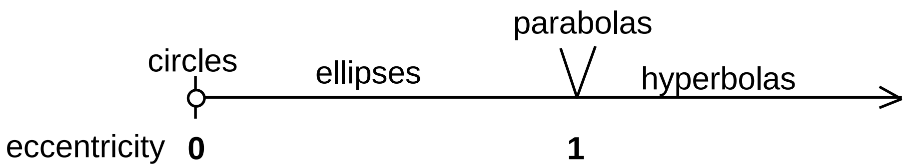
$$e = 1 \Rightarrow \textit{parabola} \;/\; 0 < e < 1 \Rightarrow \textit{ellipse} \;/\; e > 1 \Rightarrow \textit{hyperbola}$$

The Circle: As e approaches 1, the ellipse becomes more and more elongated; as it approaches zero it becomes more and more circular.
For the ellipse and hyperbola, the distance from the center to a vertex is referred to as <u>a</u> and the distance from the center to a focus as <u>c</u>. For the ellipse and hyperbola, the eccentricity e has a value of $\frac{c}{a}$.

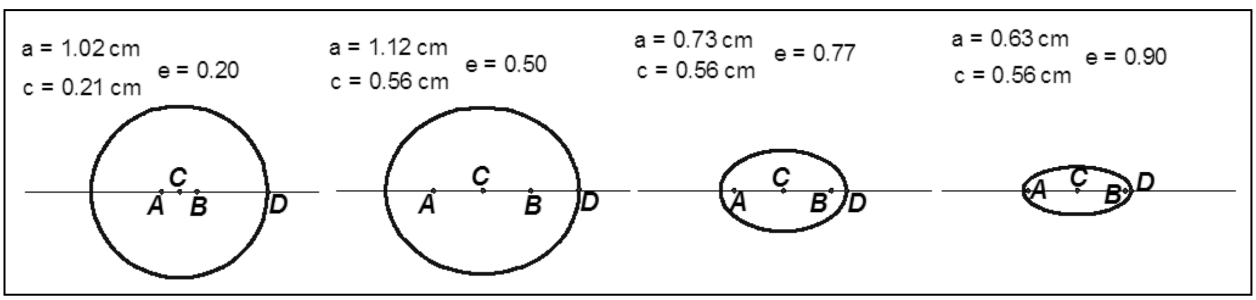
For any parabola, $e = 1$. For circles, the subject of eccentricity is a little touchy!

$PF = e \cdot d(P, \mathcal{L}) \Rightarrow e = \dfrac{PF}{d(P, \mathcal{L})}$. Since P and F are distinct points, the numerator is never zero.

However, if PF approaches zero and $d(P, \mathcal{L})$ approaches infinity, the value of the fraction approaches zero and the eccentricity approaches zero. The following graphic summarizes the relationship between eccentricity and the conic sections.



Here are the graphs of some ellipses and hyperbolas with different eccentricities
Ellipses (with a horizontal major axis):



Hyperbolas (with a horizontal major axis):