MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2014
ROUND 1 ALG 2: SIMULTANEOUS EQUATIONS AND DETERMINANTS

ANSWERS

A) ______________________________

B) (_____________ , _____________)

C) ______________________________

A) Compute the <u>smaller</u> value of A for which the determinant $\begin{vmatrix} A-2 & 3 \\ 11 & 2A+1 \end{vmatrix}$ is zero.

B) Determine the ordered pair (x, y) for which $\begin{cases} \dfrac{1}{2x}+\dfrac{7}{y}=1 \\ \dfrac{2}{x}+\dfrac{4}{y}=1 \end{cases}$.

C) Find <u>all</u> ordered triples of real numbers (a, b, c) for which $\begin{cases} a^2+8b^2-12bc=36 \\ 4ab-9c^2=36 \end{cases}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2014
ROUND 2 ALG1: EXPONENTS AND RADICALS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Given: $\{A,B,C,D\}=\{-1,2,3,4\}$
Let $N=A^B+C^D$ be a positive integer, where *A*, *B*, *C*, and *D* are distinct integers.
Compute the minimum value of *N*.

B) Find all values of *x* for which the fraction $\dfrac{1+x}{x-\dfrac{5}{2x-3}}$ is undefined.

C) Compute all values of *x* for which $\begin{cases} \dfrac{2^{x^2+Bx}}{4^{Ax}}=\dfrac{1}{32} \\ 3^{2A+B+4}=5^{2(A+B)+3}=7^{\frac{2A+5}{B}} \end{cases}$,
where *A* and *B* are rational numbers.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2014
ROUND 3 TRIGONOMETRY: ANYTHING

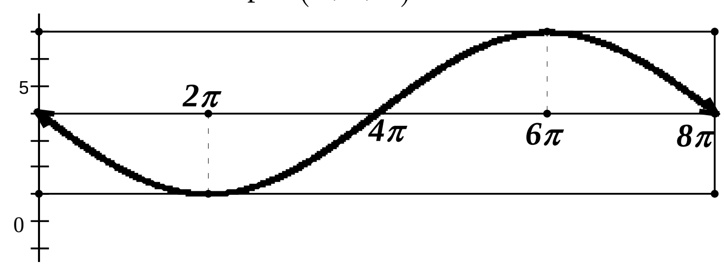
ANSWERS

A) (________ , ________ , ________)

B) ________________________________

C) (______________ , ______________)

A) The graph of $y = A\sin\left(Bx\right) + C$ is shown below.
Determine the ordered triple $\left(A, B, C\right)$.



B) Compute the degree-measure of the largest acute angle α for which $2\sin 15\alpha - 1 = 0$.

C) $\left(-1-\sqrt{3}i\right)^{100} = A\left(2^{B}\right)\left(1+\sqrt{3}i\right)$, where A and B are integers.
Compute the ordered pair $\left(A, B\right)$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2014
ROUND 4 ALG 1: ANYTHING

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Compute the absolute value of the difference between 2345_6 and 1456_7.
Express the difference in base 10.

B) The line $\mathcal{L}$ whose equation is $13x + 7y = 92$ passes through exactly one lattice point $P(x_1, y_1)$ in quadrant 1 and infinitely many lattice points in quadrants 2 and 4. Let $Q(x_2, y_2)$ be the lattice point on $\mathcal{L}$ in quadrant 2 closest to *P* and $R(x_4, y_4)$ be the lattice point on $\mathcal{L}$ in quadrant 4 closest to *P*. Compute $y_1 + y_2 + y_4$.

Recall: Lattice points are points whose coordinates are integers.

C) There were 12 contestants on "The Biggest Losers" weighing a total of 4500 lbs.
Two contestants dropped out before filming started. The average weight of the remaining contestants at the outset was 385 lbs. The average weight loss *L* for the 10 contestants who made it to the final weigh-in equals 40% of the average weight of the two contestants who dropped out. Compute *W*, the average weight of the contestants who made it to the final weigh-in.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2014
ROUND 5 PLANE GEOMETRY: ANYTHING

ANSWERS

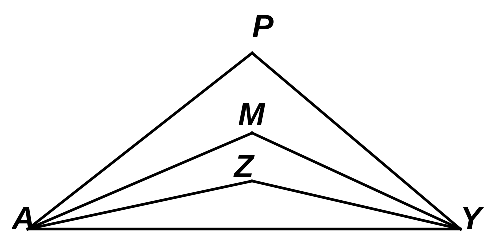
A) ______________________________

B) ______________________________

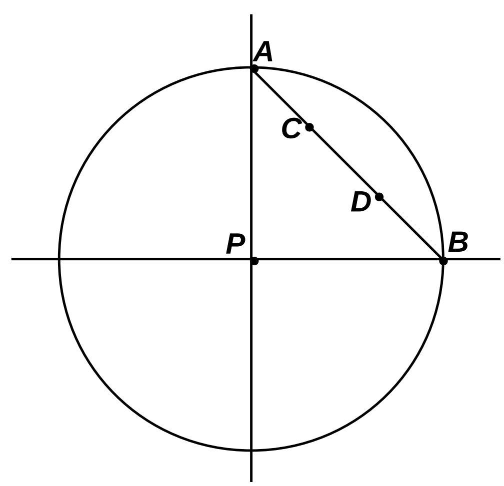
C) ______________________________

A) In quadrilateral *MIKE*, $\angle M$ is considered to be opposite $\angle K$.
In pentagon S_1US_2AN, there is no single angle which is considered opposite $\angle U$.
Only polygons with an even number of sides contain pairs of opposite angles.
Consider a 12-sided convex polygon with consecutive vertices designated $V_1, V_2, ..., V_{12}$.
Using these designations, specify a <u>3-letter name</u> for the angle opposite $\angle V_7$.

B) The trisectors of the base angles of isosceles triangle *PAY* intersect at points *M* and *Z*.
The measures of angles *M* and *Z* differ by 26°.
Compute the degree-measure of the vertex angle in ΔPAY.



C) Two points *A* and *B* lie on circle *P* with radius 6.
Initially, $m\left(\widehat{AB}\right) = 90°$, as shown in the diagram at the right. Points *C* and *D* are trisection points of chord $\overline{AB}$.
If points *A* and *B* simultaneously move around circle *P*, the locus of the trisection points is a new circle centered at *P*. If, on the other hand, point *B* is fixed and point *A* moves around circle *P*, the locus of the trisection points is two circles tangent at point *B*.
Compute the <u>sum</u> of the areas of these three circles.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2014
ROUND 6 ALG 2: PROBABILITY AND THE BINOMIAL THEOREM

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Given: $\binom{n}{r} = \dfrac{n!}{r! \cdot (n-r)!}$ and 0! = 1

Evaluate $\binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6}$

B) Find the middle term in the expansion of $\left(2A - \dfrac{k}{A}\right)^{10}$.

C) Sequences of 5 letters (repetition allowed) are to be made up, using the letters A, B, C and D. Compute the number of these sequences in which there are an even number of As, including sequences containing only Bs, Cs and Ds.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2014
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________________ D) ____________________________

B) ____________________________ E) (____________ , ____________)

C) ____________________________ F) (_____ , _____ , _____ , _____)

A) The equation $\begin{vmatrix} 2 & 4 & 8 \\ 0 & 8 & 4 \\ 0 & 1 & 2 \end{vmatrix} x^3 + \begin{vmatrix} 9 & 7 \\ -4 & 5 \end{vmatrix} x^2 - \begin{vmatrix} 7 & 10 \\ -3 & x \end{vmatrix} = 0$ has three real roots r_1, r_2, and r_3 and r_1 is a negative integer and $r_2 > r_3$. Compute the determinant $\begin{vmatrix} r_1 & r_3 \\ -r_2 & r_2 + r_3 \end{vmatrix}$.

B) Given: $Q = \dfrac{1^2 + 2^2 + 3^2 + ...n^2}{1 + 2 + 3 + ... + n}$
For how many positive integer values of n is Q an integer <u>and</u> $Q \le 2014$?

C) Compute the <u>integer</u> value of A for which $2Tan^{-1}(.4) + Tan^{-1}\left(\dfrac{1}{A}\right) = \dfrac{\pi}{4}$.

D)

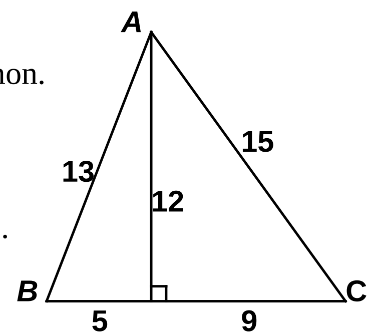
1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

1	5	9	13
2	6	10	14
3	7	11	15
4	8	12	16

The grid on the left is converted to the grid on the right by a series of additions and subtractions as follows (each of the 8 constants are nonnegative integers):
A is added to each entry in row 1 (top)
B is added to each entry in row 2
C is added to each entry in row 3
D is added to each entry in row 4
P is subtracted from each entry in column 1 (left)
Q is subtracted from each entry in column 2
R is subtracted from each entry in column 3
S is subtracted from each entry in column 4
Compute the <u>minimum</u> value of $A + B + C + D + P + Q + R + S$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2014
ROUND 7 TEAM QUESTIONS

E) A triangle with sides whose lengths are consecutive integers and which also has an integer area is not unique, but it is very uncommon. The 3-4-5 right triangle is the first example that comes to mind. The non-right triangle with the smallest perimeter (42) and smallest area (84) is shown at the right. The triangle with the next smallest perimeter P has area A. Compute the ordered pair (P, A).



F) One of my favorite T-shirts says the following:

ONLY SMART PEOPLE CAN READ THIS

It deos not mttaer in waht oredr the ltteers in a wrod are, the olny iprmoatnt tihng is taht the frist and lsat ltteer be in the rghit pclae. The rset can be a taotl mses and you can sitll raed it wouthit a porbelm. Tihs is bcuseae the bairn deos not raed ervey lteter by istlef, but the wrod as a wlohe. Taht is the phaonmneal pweor of the hmuan mnid!

The quote (starting with It) contains 72 words and I bet you can still read it.
There are thirty five words with 3 or fewer letters, fifteen 4-letter words, twelve 5-letter words and ten words with 6 or more letters.
Without changing the order of the words or the punctuation, if the letters of each word (except for the first and last) are randomly arranged, there are many versions of this T-shirt.
The probability that all the words would be spelled correctly is $\dfrac{1}{2^A 3^B 5^C 7^D}$. Determine the ordered quadruple (A, B, C, D).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2014 ANSWERS

Round 1 Alg 2: Simultaneous Equations and Determinants

A) $-\dfrac{7}{2}$ B) (4, 8) C) (6, 3, 2), (–6, –3, –2)

Round 2 Alg 1: Exponents and Radicals

A) 9 B) $-1, \dfrac{3}{2}, \dfrac{5}{2}$ C) –1, –5

Round 3 Trigonometry: Anything

A) $\left(-3, \dfrac{1}{4}, 4\right)$ or $\left(3, -\dfrac{1}{4}, 4\right)$ B) 82 C) $(-1, 99)$

Round 4 Alg 1: Anything

A) 11 B) 6 C) 255

Round 5 Plane Geometry: Anything

A) $\angle V_{12}V_1V_2$ or $\angle V_2V_1V_{12}$ B) 102 C) 40π

Round 6 Alg 2: Probability and the Binomial Theorem

-

A) 126 B) $-8064k^5$ C) 528

Team Round

A) $-\dfrac{7}{24}$ D) 36

B) 1007 E) (156, 1170)

C) 41 F) $(53, 24, 6, 2)$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2014 SOLUTION KEY

Round 1

A) $\begin{vmatrix} A-2 & 3 \\ 11 & 2A+1 \end{vmatrix} = (A-2)(2A+1) - 3\cdot 11 = 2A^2 - 3A - 35 = (2A+7)(A-5) = 0 \Rightarrow A = \cancel{5},\ \underline{-\frac{7}{2}}$

B) Let $A = \dfrac{1}{x}$ and $B = \dfrac{1}{y}$.

The given system is equivalent to $\begin{cases} \frac{1}{2}A + 7B = 1 \\ 2A + 4B = 1 \end{cases} \Leftrightarrow \begin{cases} 2A + 28B = 4 \\ 2A + 4B = 1 \ (***) \end{cases}$

Subtracting, $24B = 3 \Rightarrow B = \dfrac{1}{8} \Rightarrow y = 8$.

Substituting in (***), $2A + \dfrac{1}{2} = 1 \Rightarrow A = \dfrac{1}{4} \Rightarrow y = 4$

The required ordered pair is **__(4, 8)__**.

C) $\begin{cases} a^2 + 8b^2 - 12bc = 36 \\ 4ab - 9c^2 = 36 \end{cases}$ Subtracting, we have $a^2 - 4ab + 8b^2 - 12bc + 9c^2 = 0$.

Partitioning and sharing $8b^2$, $a^2 - 4ab \underline{+4b^2 + 4b^2} - 12bc + 9c^2 = (a-2b)^2 + (2b-3c)^2 = 0$
The sum of two non-negative quantities can only be zero if each binomial is zero!
Thus, $a = 2b = 3c$.

Substituting $a = 3c,\ b = \dfrac{3}{2}c$ in the second equation, we have

$4(3c)\left(\dfrac{3}{2}c\right) - 9c^2 = 36 \Leftrightarrow 9c^2 = 36 \Leftrightarrow c = \pm 2$

Therefore, (a, b, c) = **__(6, 3, 2)__**, **__(–6, –3, –2)__**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2014 SOLUTION KEY

Round 2

A) –1 cannot be an exponent, since this would result in $N = A^B + C^D$ being a non-integer.
So we consider $(-1)^3 + 2^4$, $(-1)^3 + 4^2$, $(-1)^4 + 2^3$, and $(-1)^4 + 3^2$
The winner (i.e. the minimum) is $(-1)^4 + 2^3 = \underline{\mathbf{9}}$.

B) Given $\dfrac{1+x}{x-\dfrac{5}{2x-3}}$, clearly $x=\dfrac{3}{2}$ causes division by 0. Considering the denominator of the complex fraction, $x-\dfrac{5}{2x-3}=\dfrac{2x^2-3x-5}{2x-3}=\dfrac{(2x-5)(x+1)}{2x-3}$ and this expression is zero for $x=-1, \dfrac{5}{2}$. Even though $(x+1)$ is a common factor and the expression simplifies to $\dfrac{2x-3}{2x-5}$, for $x=-1$, the original fraction becomes $\dfrac{0}{0}$ which is still undefined.
Thus the fraction is undefined for three values, namely, $\underline{\mathbf{-1, \dfrac{3}{2}, \dfrac{5}{2}}}$.

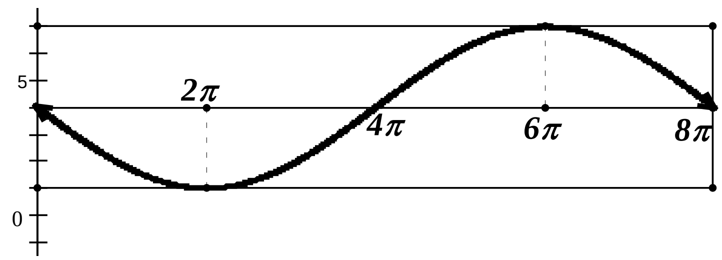
C) Given $\begin{cases} \dfrac{2^{x^2+Bx}}{4^{Ax}} = \dfrac{1}{32} \\ 3^{2A+B+4} = 5^{2(A+B)+3} = 7^{\frac{2A+5}{B}} \end{cases}$ The only way powers of 3, 5 and 7 can be equal for rational exponents would be if each exponent is zero. Therefore, $A=-\dfrac{5}{2}$ and substituting for the powers of 3 and 5, we have $B-1=0$ and $-5+2B-3=0 \Rightarrow B=1$

$$\frac{2^{x^2+Bx}}{4^{Ax}}=\frac{1}{32} \Leftrightarrow \frac{2^{x^2+Bx}}{2^{2Ax}}=2^{-5} \Leftrightarrow 2^{x^2+(B-2A)x}=2^{-5} \Rightarrow$$
$$x^2+(B-2A)x+5=0 \Leftrightarrow x^2+6x+5=(x+1)(x+5)=0 \Rightarrow x=\underline{\mathbf{-1,-5}}$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2014 SOLUTION KEY

Round 3

A)



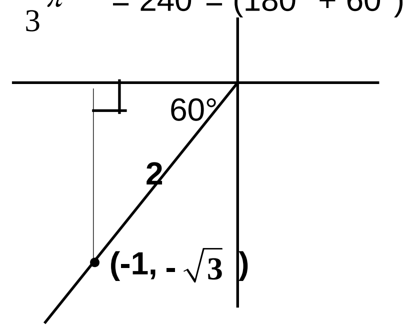
The amplitude is 3 and the graph has been flipped. This is solely controlled by the *A*-value. $A=-3$
The vertical shift is 4. This is solely controlled by the *C*-value. $C=4$
The period (which is normally 2π) has been stretched to 8π. This is solely controlled by the *B*-value.
$\dfrac{2\pi}{B}=8\pi \Rightarrow B=\dfrac{1}{4}$. Thus, $(A,B,C)=\underline{\left(\mathbf{-3,\frac{1}{4},4}\right)}$ or $\underline{\left(\mathbf{3,-\frac{1}{4},4}\right)}$.

B) $\sin 15\alpha=\dfrac{1}{2}\Leftrightarrow 15\alpha=\begin{cases}30+360n\\150+360n\end{cases}\Rightarrow \alpha=\begin{cases}2+24n\\10+24n\end{cases}\Rightarrow \alpha=\begin{cases}2,26,50,74\\10,34,58,\underline{\mathbf{82}}\end{cases}$

C) $\left(-1-\sqrt{3}i\right)^{100}=\left(2cis\dfrac{4\pi}{3}\right)^{100}=2^{100}cis\dfrac{400\pi}{3}=2^{100}cis\left(\left(132+\dfrac{4}{3}\right)\pi\right)=2^{100}cis\dfrac{4\pi}{3}$

$=2^{100}\left(-\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i\right)=\left(-\dfrac{1}{2}\right)2^{100}\left(1+\sqrt{3}i\right)=(-1)2^{99}\left(1+\sqrt{3}i\right)\Rightarrow (A,B)=\underline{(\mathbf{-1,99})}$.

$\dfrac{4}{3}\pi^{rad}$ = 240°= (180° + 60°)



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2014 SOLUTION KEY

Round 4

A) 2345_6 equals $2(6)^3 + 3(6)^2 + 4(6) + 5$ equals 569 and
1456_7 equals $1(7)^3 + 4(7)^2 + 5(7) + 6$ equals 580.
Thus, the difference is **11**.

B) $y = \dfrac{92-13x}{7} = 13 - x + \dfrac{1-6x}{7}$

x must be picked so that $\dfrac{1-6x}{7}$ produces an integer.

$x = -1$ produces a multiple of 7 in the numerator and other possible values of x can be found by adding or subtracting 7 (or any multiple of 7).

$x = -1 \Rightarrow y = 15$

The slope of the line is $\dfrac{-13}{7}$, so as x increases by 7, y decreases by 13 or equivalently as x decreases by 7, y increases by 13.

Thus, we have ordered pairs (–1, 15), (6, 2), (13, –11). P must be (6, 2).

Other points in quadrants 2 and 4 will be further from point P.

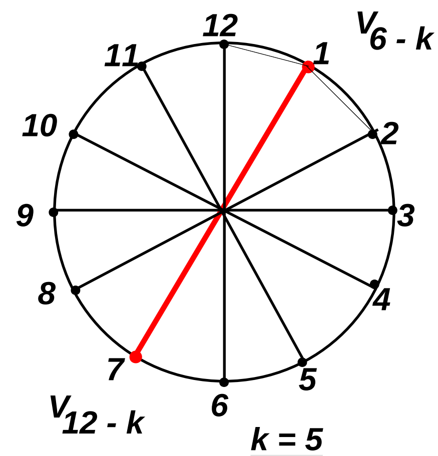
Therefore, $y_1 + y_2 + y_4 = 2 + 15 + (-11) = \underline{\mathbf{6}}$.

C) Let x and y denote the weights of the 2 contestants who initially dropped out.

$\dfrac{4500-(x+y)}{10} = 385 \Rightarrow x + y = 650 \Rightarrow \text{Ave} = \dfrac{x+y}{2} = 325$ Then:

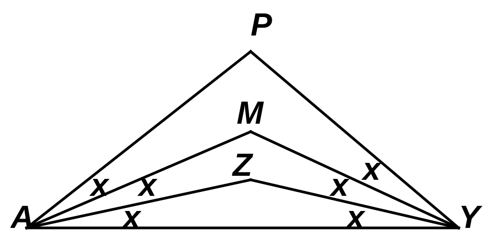
$L = \dfrac{3850-10W}{10} = 0.4(325) \Rightarrow L = 385 - W = 130 \Rightarrow W = \underline{\mathbf{255}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2014 SOLUTION KEY

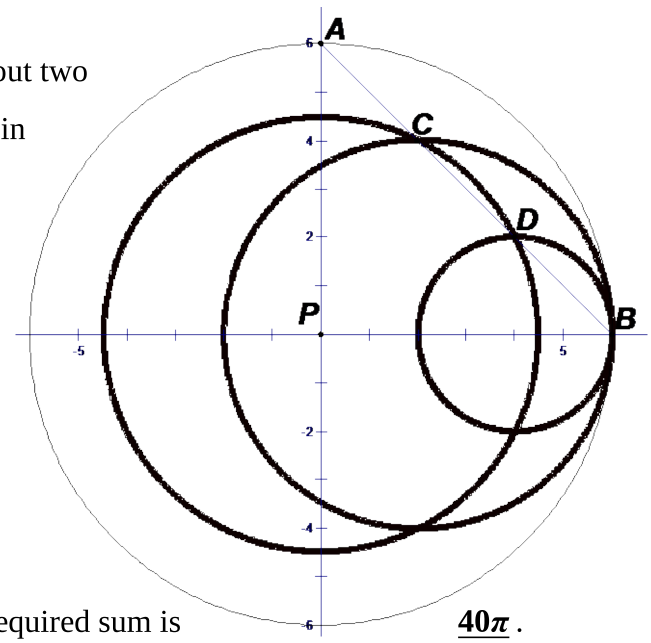


Round 5

A) Consider a clock face. 12 is opposite 6. 11 is opposite 5. 10 is opposite 4. In general, $12-k$ is opposite $6-k$.
$12-k=7 \Rightarrow k=5 \Rightarrow 6-k=1$.
Therefore, V_1 is opposite V_7 and the adjacent vertices are V_{12} and V_2, resulting in 3-letter names of either $\underline{\angle \mathbf{V_{12}V_1V_2}}$ or $\underline{\angle \mathbf{V_2V_1V_{12}}}$ for the opposite angle.



B) Let P be the vertex angle.
Examining the diagram at the right,
$m\angle P = 180 - 6x$
$m\angle M = 180 - 4x$
$m\angle Z = 180 - 2x$
Since M and Z differ by 26°, x must be 13.
Therefore, $m\angle P = 180 - 6\cdot 13 = \underline{\mathbf{102}}°$



C) With point B fixed and A moving, C and D trace out two circles with radii $\frac{1}{3}\cdot 6 = 2$ and $\frac{2}{3}\cdot 6 = 4$, resulting in areas of 4π and 16π .

If A and B are simultaneously set in motion, we get a circle concentric with the original.
Its radius is PD (or PC).
$PB = 6 \Rightarrow AB = 6\sqrt{2} \Rightarrow DB = 2\sqrt{2}$ and
$m\angle PBA = 45°$. Using the Law of Cosines on ΔDPB, we have

$$PD^2 = \left(2\sqrt{2}\right)^2 + 6^2 - 2\cdot 2\sqrt{2}\cdot 6\cdot \cos 45°$$
$$= 8 + 36 - 24 = 20$$

Thus, the area of the third circle is 20π and the required sum is $\underline{\mathbf{40\pi}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2014 SOLUTION KEY

Round 6

A) Appealing strictly to the definition, $\binom{7}{1}=\binom{7}{6}$, $\binom{7}{2}=\binom{7}{5}$ and $\binom{7}{3}=\binom{7}{5}$, so only three combinations need be evaluated.

$\binom{7}{1}=\frac{7!}{1!\cdot 6!}=\frac{7\cdot\cancel{6!}}{\cancel{1!}\cdot\cancel{6!}}=7$ $\quad\binom{7}{2}=\frac{7!}{2!\cdot 5!}=\frac{7\cdot 6\cdot\cancel{5!}}{2\cdot\cancel{5!}}=21$ $\quad\binom{7}{3}=\frac{7!}{3!\cdot 4!}=\frac{7\cdot\cancel{6}\cdot 5\cdot\cancel{4!}}{\cancel{6}\cdot\cancel{4!}}=35$

Thus, the required sum is $2(7+21+35)=\underline{\mathbf{126}}$.

Alternate solution #2:
These are the numbers from the 7th row of Pascal's triangle, excluding the first and last, both of which are 1s. All the numbers in the 7th row add up to $2^7=128$. Thus, our total is $128-2=\underline{\mathbf{126}}$.

Alternate Solution #3
According to the binomial Theorem,

$$(a+b)^7=\binom{7}{0}a^7+\binom{7}{1}a^6b^1+\binom{7}{2}a^5b^2+\ldots+\binom{7}{6}a^1b^6\binom{7}{7}b^7$$

The exponents of *a* go down by 1. The exponents of *b* go up by 1.
The exponent of the *b*-term matches the bottom number in the combination.
If we let $a=b=1$, none of this matters.

We have simply $2^7=\binom{7}{0}+\boxed{\binom{7}{1}+\ldots+\binom{7}{6}}+\binom{7}{7}$.

The boxed quantity is the required sum and we have the same result.

B) Since there are 11 terms in the expansion, the middle term is the 6th term.

$$\Rightarrow\binom{10}{5}(2A)^5\left(-\frac{k}{A}\right)^5=-\frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot 32A^5\cdot\frac{k^5}{A^5}=$$

$$-\frac{\overset{2}{\cancel{10}}\cdot 9\cdot\overset{2}{\cancel{8}}\cdot 7\cdot\cancel{6}}{\cancel{1\cdot 2\cdot 3}\cdot\cancel{4}\cdot\cancel{5}}\cdot 32\cancel{A^5}\cdot\frac{k^5}{\cancel{A^5}}=-252(32)k^5=\underline{\mathbf{-8064k^5}}$$

C) There are 3^5 = <u>243</u> sequence with no As. There are $\binom{5}{2}=10$ ways to position 2 As.

(_1_ _2_ _3_ _4_ _5_) $\Rightarrow$ Positions: 12, 13, 14, 15, 23, 24, 25, 34, 35, 45
Now fill in the remaining positions with any of the other letters.

Pick two letters and place them in the remaining 3 positions: $\binom{5}{2}3^3=10(27)=\underline{270}$

There are $\binom{5}{4}=5$ ways to arrange 4 As and 3 choices for the 5th position $\Rightarrow$ <u>15</u>

Thus, an even number of As may occur 243 + 270 + 15 = **<u>528</u>** ways.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2014 SOLUTION KEY

Team Round

A) $\begin{vmatrix} 2 & 4 & 8 \\ 0 & 8 & 4 \\ 0 & 1 & 2 \end{vmatrix} x^3 + \begin{vmatrix} 9 & 7 \\ -4 & 5 \end{vmatrix} x^2 - \begin{vmatrix} 7 & 10 \\ -3 & x \end{vmatrix} = 0$ Evaluate $\begin{vmatrix} r_1 & r_3 \\ -r_2 & r_2 + r_3 \end{vmatrix}$.

The lead coefficient is easily evaluated since the left column is all zeros except the first entry. Reducing the determinant of a square $n \text{ x } n$ matrix to a series of determinants of $(n-1) \text{ x } (n-1)$ matrices is called expansion by minors.

$$\begin{vmatrix} 2 & 4 & 8 \\ 0 & 8 & 4 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} \boxed{2} & \cancel{4} & \cancel{8} \\ \cancel{0} & 8 & 4 \\ \cancel{0} & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 8 & 4 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 4 & 8 \\ 1 & 2 \end{vmatrix} + 0\begin{vmatrix} 4 & 8 \\ 8 & 4 \end{vmatrix} = 2(8 \cdot 2 - 1 \cdot 4) = 24$$

The quadratic coefficient is simply $9 \cdot 5 + 4 \cdot 7 = 73$.
The last determinant is $7 \cdot x + 3 \cdot 10 = 7x + 30$
Thus, the cubic equation is $24x^3 + 73x^2 - 7x - 30 = 0$.
The determinant to be evaluated is $r_1r_2 + r_1r_3 + r_2r_3$.
We could certainly use the clues to factor the cubic expression and then plug in the roots and evaluate the expression. But none of that is necessary!! Notice that when $(x - r_1)(x - r_2)(x - r_3) = 0$ is expanded, you get $x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3 = 0$.
If the lead coefficient were 1, the expression we seek would be just the coefficient of the *x*-term.
So dividing both sides of our cubic by 24, we get the required determinant, namely $\underline{\mathbf{-\frac{7}{24}}}$.

Here's a check: $24x^3 + 73x^2 - 7x - 30 = (x+3)(8x-5)(3x+2) = 0$

$\Rightarrow (r_1, r_2, r_3) = \left(-3, \frac{5}{8}, -\frac{2}{3},\right)$ and

$$r_1r_2 + r_1r_3 + r_2r_3 = \left(-3 \cdot \frac{5}{8}\right) + \left(-3 \cdot -\frac{2}{3}\right) + \left(\frac{5}{8} \cdot -\frac{2}{3}\right) = \frac{-15}{8} + 2 + \frac{-10}{24} = \frac{-45 + 48 - 10}{24} = -\frac{7}{24}$$

B) We know that $1^2 + 2^2 + 3^2 + \ldots n^2 = \frac{n(n+1)(2n+1)}{6}$ and $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

Thus, $Q = \frac{1^2 + 2^2 + 3^2 + \ldots n^2}{1 + 2 + 3 + \ldots + n} = \frac{n(n+1)(2n+1)}{6} \cdot \frac{2}{n(n+1)} = \frac{2n+1}{3}$

Q is an integer for n = 1, 4, 7, … $(3k - 2)$, where k is a positive integer.

$\frac{2(3k-2)+1}{3} \le 2014 \Leftrightarrow k \le \frac{6042+3}{6} = 1007.5$

Thus, Q is an integer for **<u>1007</u>** values of n.

C) Let $\alpha = Tan^{-1}(.4)$ and $\beta = Tan^{-1}\left(\frac{1}{A}\right)$. Taking tan of both sides, $\tan\left(2\alpha+\beta\right)=1$. Expanding,

$$\frac{\tan 2\alpha + \tan\beta}{1-\tan 2\alpha\cdot\tan\beta} = \frac{\tan 2\alpha + \frac{1}{A}}{1-\tan 2\alpha\cdot\frac{1}{A}} = 1 \Rightarrow \tan 2\alpha + \frac{1}{A} = 1 - \tan 2\alpha\cdot\frac{1}{A} \Rightarrow \tan 2\alpha\left(1+\frac{1}{A}\right) = 1-\frac{1}{A}$$

$\Rightarrow \tan 2\alpha = \dfrac{1-\frac{1}{A}}{1+\frac{1}{A}} = \dfrac{A-1}{A+1}$. We know that $\tan\alpha = 0.4 = \dfrac{2}{5} \Rightarrow \tan 2\alpha = \dfrac{2\left(\frac{2}{5}\right)}{1-\left(\frac{2}{5}\right)^2} = \dfrac{20}{25-4} = \dfrac{20}{21}$

Therefore, $\dfrac{A-1}{A+1} = \dfrac{20}{21} \Rightarrow 21A - 21 = 20A + 20 \Rightarrow A = \underline{41}$.

FYI: $Tan^{-1}(.4) \approx 21.80°$ and $Tan^{-1}\left(\frac{1}{41}\right) \approx 1.40°$ and $2\left(21.80\right)+1.40 = 45° = \dfrac{\pi}{4}^{\,rad}$

Alternately, start with $\tan\alpha = 0.4 = \dfrac{2}{5}$ and get $\tan 2\alpha = \dfrac{20}{21}$ as above. Rearrange the terms in the equation and take the tangent of both sides $\left(\text{Recall}: \tan\left(\alpha-\beta\right) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}\right)$:

$$2Tan^{-1}\left(.4\right) + Tan^{-1}\left(\frac{1}{A}\right) = \frac{\pi}{4} \Leftrightarrow Tan^{-1}\left(\frac{1}{A}\right) = \frac{\pi}{4} - 2Tan^{-1}\left(.4\right) \Rightarrow \frac{1}{A} = \frac{1-\frac{20}{21}}{1+1\cdot\frac{20}{21}} = \frac{1}{41} \Rightarrow A = \underline{\mathbf{41}}.$$

D)

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

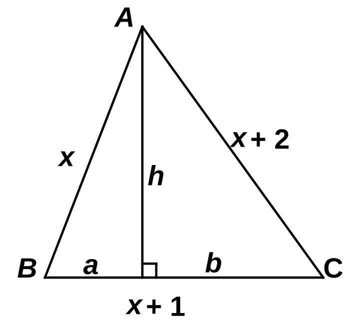
A/P	Q	R	S
B			
C			
D			

1	5	9	13
2	6	10	14
3	7	11	15
4	8	12	16

The entries along the main diagonal (upper left to lower right) are equal.
Unless what we add to row 1, we also subtract from column 1, the entry in the upper left corner of the resulting grid will change. Therefore, $A = P$.
Similarly, $B = Q$, $C = R$ and $D = S$.
Since 2 in the original grid is replaced by 5, what we add to row 1 must be 3 more than what we subtract from column 2. Thus, $A = Q + 3$.
Similarly, by examining the entries in the top row, $A = R + 6$, $A = S + 9$
To minimize the sum, let $S = 0 \Rightarrow D = 0$, $A = P = 9$, $R = C = 3$, $Q = B = 6$.
The minimum sum is $A+B+C+D+P+Q+R+S$ = **<u>36</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2014 SOLUTION KEY

E) Assume the triangle we seek has sides as indicated at the right.



Solution #1: (using Heron's Formula)

$s = \dfrac{3x+3}{2}$ $A = \sqrt{\dfrac{3(x+1)}{2} \cdot \dfrac{x+3}{2} \cdot \dfrac{x+1}{2} \cdot \dfrac{x-1}{2}}$

$\Rightarrow\ 4A = (x+1)\sqrt{3(x+3)(x-1)}$

By trial and error, $x = 3 \Rightarrow 4A = 4\sqrt{3 \cdot 6 \cdot 2} = 24 \Rightarrow A = 6$ (This is the 3-4-5 triangle.)

$x = 13 \Rightarrow 4A = 14\sqrt{3 \cdot 16 \cdot 12} = 14 \cdot 24 \Rightarrow A = 84$ (This is the 13-14-15 triangle above.)

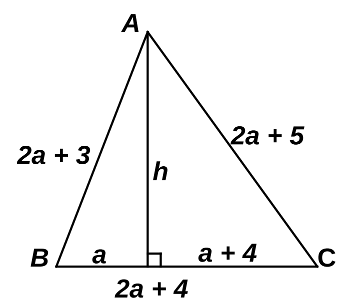
$x = 51 \Rightarrow 4A = 52\sqrt{3 \cdot 54 \cdot 50} = 52 \cdot \sqrt{81 \cdot 100} = 52 \cdot 90 \Rightarrow A = 1170$

We leave it to you to check that integer values of *x* between 13 and 51 do not yield any solutions.

Thus, the next non-right triangle is a 51-52-53 triangle and $(P, A) =$ **(156,1170)**

Solution #2 (Pythagorean Theorem, Quadratic Formula and Recognition of Perfect squares)

$h^2 = x^2 - a^2 = (x+2)^2 - b^2 = (x+2)^2 - (x+1-a)^2$



Expanding,

$x^2 - a^2 = x^2 + 4x + 4 - \left(x^2 + 1 + a^2 + 2x - 2ax - 2a\right)$

$x^2 - a^2 = (2+2a)x - \left(a^2 - 2a - 3\right)$

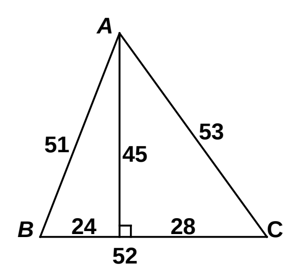
$x^2 - (2+2a)x - (2a+3) = 0$ Applying the quadratic formula, we have

$x = \dfrac{2(a+1) \pm \sqrt{4(a+1)^2 + 4(2a+3)}}{2} = (a+1) \pm \sqrt{a^2 + 4a + 4}$

$= (a+1) \pm (a+2) = 2a+3$ or $\cancel{-1}$

Substituting for *x*, the diagram has dimensions strictly in terms of *a*.
Equating the expressions for *h*,

$h = \sqrt{(2a+3)^2 - a^2} = \sqrt{3a^2 + 12a + 9}$



We can look at the radicand as $3(a+3)(a+1)$ or $3\left((a+2)^2 - 1\right)$

Either way it must be perfect square.

Taking the second view, $(a+2)^2 - 1$ must be three times a perfect square, so

$(a+2)^2 - 1 = 3k^2$ or $a = -2 + \sqrt{3k^2 + 1}$

Choosing *k* so that $3k^2 + 1$ is a perfect square (greater than 4), we have solutions.

$k = 4 \Rightarrow a = -2 + 7 = 5$ and we have the given solution.

$k = 5 \ldots 15 \Rightarrow$ 76, 104, 148, 193, 244, 301, 364, 433, 508, 589, 676 = 26^2

Here are the relevant perfect squares (11^2 thru 25^2) for comparison:

121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625

Thus, $5 \le k \le 14$ fail.

E) continued

$k = 15 \Rightarrow a = -2 + \sqrt{676} = -2 + 26 = 24$ and we have this solution

The area is $\frac{1}{2} \cdot 45 \cdot 52 = 45 \cdot 26 = 1170$. Thus, $(P, A) = \underline{(\mathbf{156,1170})}$.

Additional triangles occur for

$k = 56 \Rightarrow a = 95$ (193-194-195 / altitude 168)

$k = 209 \Rightarrow a = 360$ (723-724-725 / altitude 627)

The next value of k "discovered" by the TI-84+ is $k = 362$, but then $3k^2 + 1$ would end in 3.

How many perfect squares do you know ending in 3?

Oops, the TI-84+ has exceeded its limits, again!

What is the next k-value?

F)

> It deos not mttaer in waht oredr the ltteers in a wrod are, the olny iprmoatnt tihng is taht the frist and lsat ltteer be in the rghit pclae. The rset can be a taotl mses and you can sitll raed it wouthit a porbelm. Tihs is bcuseae the bairn deos not raed ervey lteter by istlef, but the wrod as a wlohe. Taht is the phaonmneal pweor of the hmuan mnid!

The thirty-five words of 1, 2 or 3 letters can be ignored.

The arrangement of the letters in each of the fifteen 4-letter words can be done in 2 ways, since no interior letters are duplicated. $\Rightarrow 2^{15}$

5-letter words (12): order/thing/first/right/place/total/still/brain/every/whole/power/human

The arrangement of the letters in all these 5-letter words can be done in 6 ways, since no interior letters are duplicated. $\Rightarrow 6^{12} = 2^{12} \cdot 3^{12}$

There are four 6-letter words (two separate occurrences of the word 'letter')

itself $\left[4! = 2^3 3\right]$ matter $\left[\frac{4!}{2!} = 12 = 2^2 3\right]$ letter $\left[\left(\frac{4!}{2!2!}\right)^2 = (3!)^2 = 2^2 3^2\right]$

There are four 7 letter words

letters $\left[\frac{5!}{2!2!} = 2 \cdot 3 \cdot 5\right]$ without/problem/because $\left[(5!)^3 = \left(2^3 \cdot 3 \cdot 5\right)^3 = 2^9 3^3 5^3\right]$

There is one 9-letter word – important $\left[7! = 2^4 3^2 5^1 7^1\right]$

There is one 10-letter word – phenomenal $\left[\frac{8!}{2!2!} = 2 \cdot 7! = 2^5 3^2 5^1 7^1\right]$

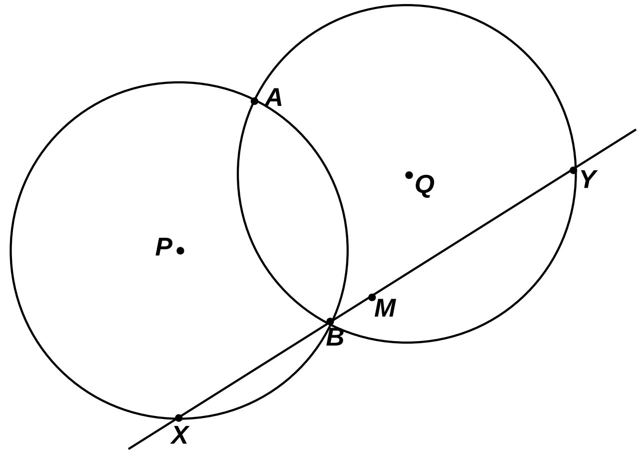
$\Rightarrow 2^{15+12+3+2+2+1+9+4+5} \cdot 3^{12+1+1+2+1+3+2+2} \cdot 5^{1+3+1+1} \cdot 7^{1+1} = 2^{53} 3^{24} 5^6 7^2$

Thus, $(A, B, C, D) = \underline{(\mathbf{53, 24, 6, 2})}$

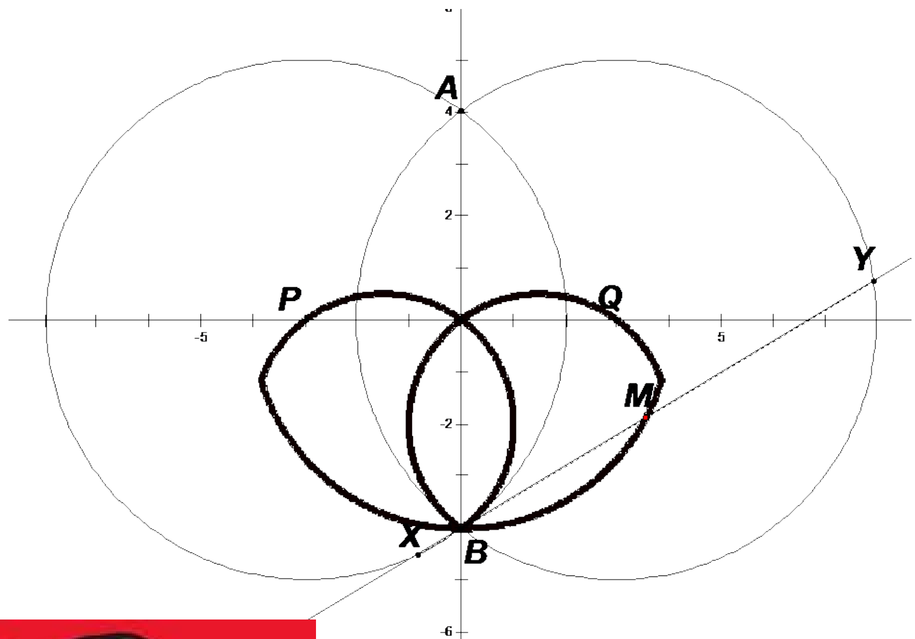
I always thought this T-shirt was 1 in a million, but it's actually more like

1 in 1 950 000 000 000 000 000 000 000 000 000 000 (or 1 : 1.95 x 10^{33}) .

Honesty is the best policy!
Here is the original problem 5C)
Two circles with centers P and Q each with a radius of 5 intersect at points A and B. $PQ = 6$.
An arbitrary line through point B intersects the two circles at points X and Y. Consider the locus of point M, the midpoint of segment $\overline{XY}$.
Compute the maximum value of PM.



Unfortunately the locus is not a circle as I originally conjectured.
Here's a sketch of the actual locus of point M as an arbitrary line through point B rotates through 360° with $PQ = 6$ and $PB = QB = 5$. I did not expect this at all.



Sort of reminds me of this character.
It's stretch, but I guess I'm hungry.

If anyone makes any progress on my original question, please send it to **olson.re@gmail.com**.

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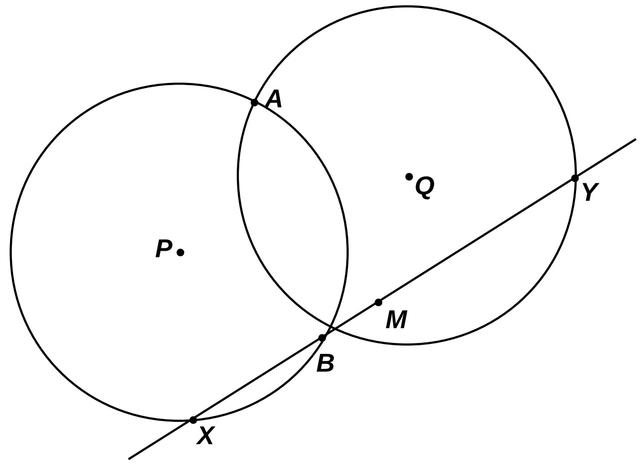

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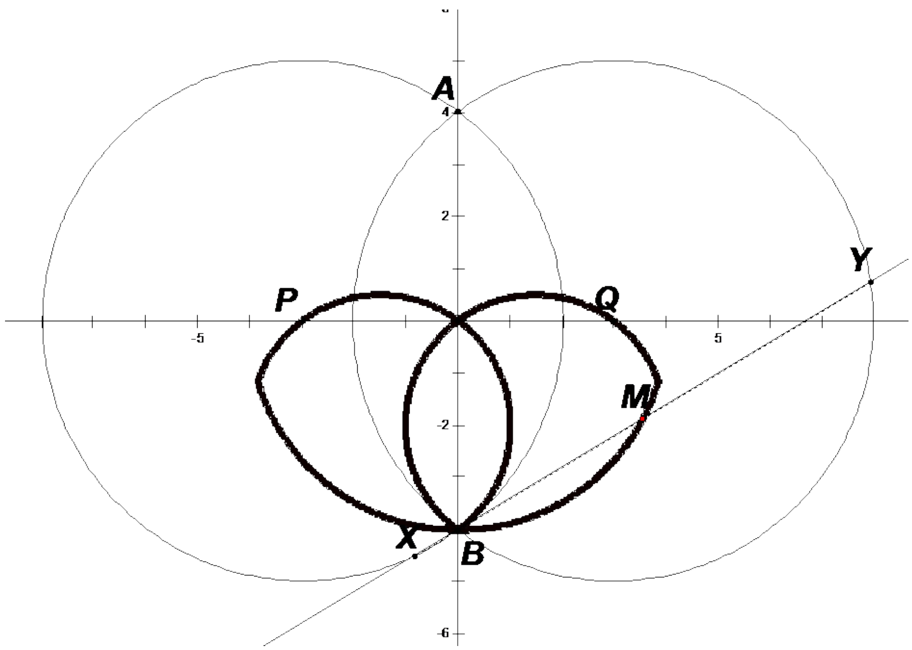

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