MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2014
ROUND 1 ALGEBRA 2: ALGEBRAIC FUNCTIONS

ANSWERS

A) ________________________________

B) ________________________________

C) (_________ , _________ , _________)

A) Given: $f(x) = -7$, $g(x) = x^2 + 4$ and $h(x) = -2x$

Compute $f\left(g\left(h(-3)\right)\right) + g\left(f\left(h(2014)\right)\right) + h\left(g(-3)\right)$.

B) Given: $y = f(x)$ is a linear function and $f\left(f(x)\right) = 4x + 15$

Compute <u>all</u> possible values of $f\left(2\right)$.

C) Given: $f(x) = 2x^3 - 3x^2 + 8x - 1$

For a specific value of h, $f(x+h) = 2x^3 + Bx + C$, that is, the quadratic term disappears.

Compute the ordered triple $\left(h, B, C\right)$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2014
ROUND 2 ARITHMETIC / NUMBER THEORY

ANSWERS

A) (____________ , ____________)

B) ______________________________

C) ______________________________

A) A teenager thought to himself, "Two years ago I was *in my prime***, but two years from now I won't be". Let M and m denote the maximum and minimum number of years until I am no longer a teenager. Determine the ordered pair (M, m).
** Translation: My age (in years) was a prime number.

B) Determine <u>all</u> integers between 2014 and 2140 which are multiples of both 6 and 21.

C) Exactly three of the following numbers are the squares of primes:
3027 4489 5329 7921 9025
Compute the sum of these 3 primes.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2014
ROUND 3 TRIG: IDENTITIES AND/OR INVERSE FUNCTIONS

ANSWERS

A) (_____________ , _____________)

B) ______________________________

C) ______________________________

A) Given: $$\cos\left(Sin^{-1}(-.5)\right)+\cot\left(Tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right)+Cos^{-1}\left(\cos\frac{\pi}{6}\right)=\frac{\pi-A}{B}$$

Compute the ordered pair (A,B).

B) Compute <u>all</u> values of x (in degrees), where $0°\le x<360°$, that satisfy

$$\cos(x+150°)+\cos(x+30°)=\cos(290°)$$

C) Compute: $$\sin\left(2Sin^{-1}\left(-\frac{4}{5}\right)+\frac{\pi}{2}\right)$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2014
ROUND 4 ALG 1: WORD PROBLEMS

ANSWERS

A) ______________________ minutes

B) ______________________________

C) (____________ , ____________)

A) Bonnie and Clyde can pull off a job together in 6 minutes. Thanks to Title 9, Bonnie can do three times as much of the job as Clyde can in any given amount of time.
How long (in minutes) will it take Bonnie to do the job alone?

B) A true story.
Not too long ago, my teenage son's age equaled his shoe size.
Fortunately, his growth slowed down. Now, 26 years later, he is 6' 5" tall, but his shoe size is only 1 size larger and only equals $\frac{3}{8}$ of his current age. Compute his current shoe size.

C) Last March, the stats for RightStuff High School were as follows:
The unique <u>mode</u> of their 7 round scores was 18. This score occurred twice.
The <u>median</u> of their 7 round scores was 22.
Their lowest round score (also unique) was in the team round.
Let M and m denote our maximum and minimum possible score for the meet.
Compute (M, m).

A Reminder (as if you didn't already know):
MML competition consists of 7 rounds, 6 individual and a team round.
There are 3 questions in each individual round
In individual rounds, 5 mathletes compete.
In the team round, all 10 mathletes collaborate.
The maximum score in any individual round is 30 points – 2 pts. per correct answer
The maximum score in the team round is 18 points – 3 pts. per correct answer.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2014
ROUND 5 PLANE GEOMETRY: CIRCLES

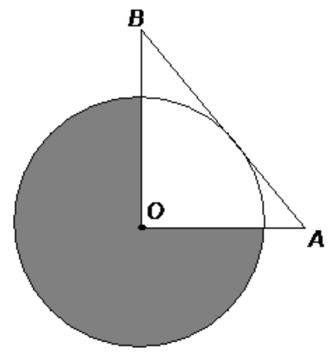
ANSWERS

A) ________________________________

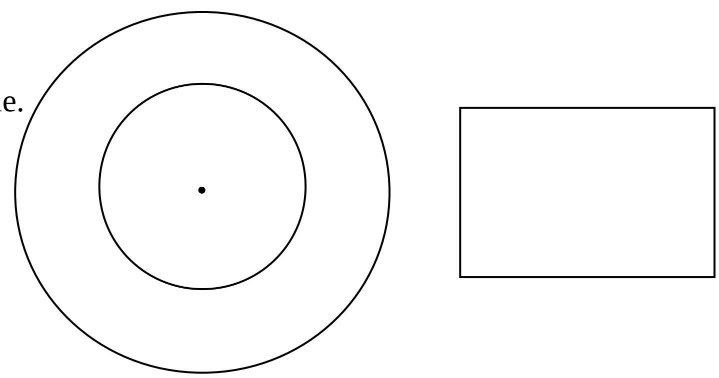
B) ________________________________

C) ________________________________

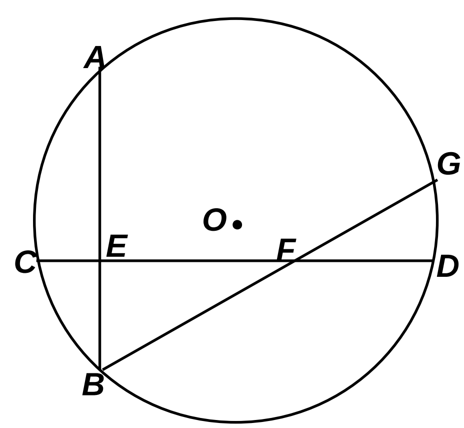
A) ΔBOA has sides of length 3, 4 and 5.
Circle O is tangent to $\overline{AB}$.
Compute the area of the shaded region.



B) The smaller of two concentric circles has a <u>diameter</u> that is 2/3 the <u>radius</u> of the larger circle.
The area of the region between the circles is equal to the area of a rectangle whose length is twice its width.
Find the ratio of the length of the rectangle to the diameter of the larger circle.



C) In circle O, $\overline{AB}$ and $\overline{CD}$ are perpendicular chords.
$CE = 2$, $DE = 20$, $BF = 13$
AE and BE are integers, where $AE > BE$ and chord $\overline{AB}$ is as short as possible.
Compute FG.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2014
ROUND 6 ALG 2: SEQUENCES AND SERIES

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) $(A,B,C)=(9,17,25)$ is an arithmetic sequence of positive integers.
If B is reduced by x, then (A,B,C) becomes a geometric sequence of <u>positive</u> integers, whose common multiplier is R. Compute R.

B) Given: $1+3+6+10+...+\frac{n(n+1)}{2}=\frac{n(n+1)(n+2)}{6}$ for any positive integers n.
If twice as many terms were added (that is, n is doubled), the sum would be multiplied by a factor of 7. Compute n, if $n \neq 0$.

C) The first five terms in a sequence of random numbers are 7, 16, 13, 4, and A.
If these integers are sorted, the median and the mean are equal.
A may be <u>any</u> integer - positive, negative or zero.
Compute <u>all</u> possible values of A.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2014
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________________ D) ____________________________

B) ____________________________ E) ____________________________

C) ____________________________ F) ____________________________

A) Let $y = f(x)$ define a 5th degree polynomial function. If $y = f(x)$ is an <u>odd</u> function and $f(1) = 0$, $f(2) = 42$ and $f\left(\frac{1}{2}\right) = \frac{3}{16}$, compute the zeros of $y = f(x)$.

B) Five boxes of apples (each with integral weights) are weighed two at a time. Every possible combination is weighed once, and the weights of each pair, in increasing order are:

110, 112, 113, 114, 115, 116, 117, 118, 120, 121 (111 and 119 are missing)

If (54, a, b, c, 62) denote the individual weights of the five boxes, where $54 < a < b < c < 62$. Determine <u>how many</u> distinct ordered triples (a, b, c) are possible.

C) Specify <u>all</u> real numbers for which the expression $Sin^{-1}\left(6x^2 - 5x\right)$ is defined.

D) One of my favorite hymns ("Holy, Holy, Holy") contains 16 measures. In $\frac{4}{4}$ time, each measure gets 4 beats and a quarter note is the unit, getting 1 beat. The hymn contains 5 different types of notes:

<u>Q</u>uarter note (Q) – 1 beat

<u>H</u>alf note (W) – 2 beats

<u>W</u>hole note (o) - 4 beats

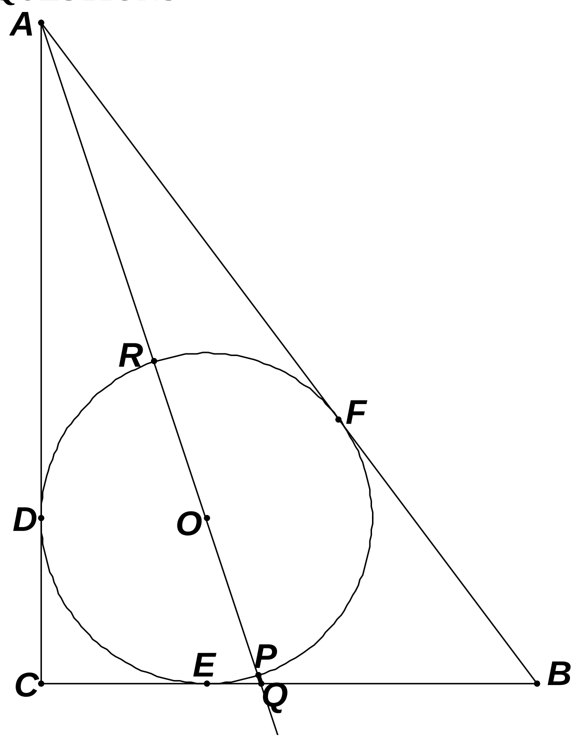
<u>D</u>otted quarter (Q.) - $1\frac{1}{2}$ beats

<u>E</u>ighth note (E) - $\frac{1}{2}$ beat

The hymn contains a total of 48 notes. The ratio of the number of quarter notes to the number of eighth notes is 7 : 1. There are as many dotted quarter notes as eighth notes. If the quarter notes account for as many beats as the half and whole notes combined, how many half notes are there?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2014
ROUND 7 TEAM QUESTIONS

E) Circle O is inscribed in 3-4-5 right ΔABC.
Specifically, $BC = 3$ and $AC = 4$.
$\overrightarrow{AO}$ intersects circle O in points P and R and intersects $\overline{BC}$ in point Q.
The ratio $AO : PQ = k : 1$.
Compute k.



F) Let P_n be a sequence of polynomials defined by $P_n(x) = xP_{n-1}(x) - P_{n-2}(x)$, where $P_0(x) = 1$ and $P_1(x) = 1$. Compute $\sum_{n=0}^{n=6} P_n(3)$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2014– ANSWERS

Round 1 Alg 2: Algebraic Functions

A) 20

B) –19, 9 (in any order)

C) $(0.5, 6.5, 2.5)$

Round 2 Arithmetic/ Number Theory

A) (7, 1)

B) 2016, 2058, 2100 (in any order)

C) 229

Round 3 Trig Identities and/or Inverse Functions

A) $\left(3\sqrt{3}, 6\right)$

B) 200, 340

C) $-\frac{7}{25}$

Round 4 Alg 1: Word Problems

A) 8

B) 15

C) (157, 136)

Round 5 Geometry: Circles

A) $\frac{108\pi}{25}$ (or 4.32π)

B) $\frac{2\sqrt{\pi}}{3}$

C) $\frac{112}{13}$

Round 6 Alg 2: Sequences and Series

A) $\frac{5}{3}$

B) 10

C) –5, 10, 25

Team Round

A) $0, \pm 1, \pm \frac{\sqrt{2}}{2}$

B) 1

C) $-\frac{1}{6} \le x \le \frac{1}{3}$ or $\frac{1}{2} \le x \le 1$
or $\left[-\frac{1}{6}, \frac{1}{3}\right], \left[\frac{1}{2}, 1\right]$

D) 10

E) $30 + 9\sqrt{10}$ or $3\left(10 + 3\sqrt{10}\right)$

F) 145

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2014 SOLUTION KEY

Round 1

A) Notice the first term requires no evaluation and the arithmetic challenge in the second term may be avoided, since the function *f* returns –7, regardless of its argument.

$f(x) = -7,\ g(x) = x^2 + 4$ and $h(x) = -2x$

$f\left(g\left(h(-3)\right)\right) + g\left(f\left(h(2014)\right)\right) + h\left(g(-3)\right) = -7 + g(-7) + h(13) = -7 + 53 - 26 = \underline{\mathbf{20}}$

B) Let $f(x) = mx + b$. Then: $f\left(f(x)\right) = m(mx+b) + b = m^2x + b(m+1) = 4x + 15$.

Thus, $m^2 = 4$ and $b(m+1) = 15$

$m = 2 \Rightarrow 3b = 15 \Rightarrow b = 5 \Rightarrow y = 2x + 5 \Rightarrow f(2) = \underline{\mathbf{9}}$.

$m = -2 \Rightarrow b = -15 \Rightarrow y = -2x - 15 \Rightarrow f(2) = \underline{\mathbf{-19}}$.

C) $f(x+h) = 2(x+h)^3 - 3(x+h)^2 + 8(x+h) - 1$

$= 2\left(x^3 + 3x^2h + 3xh^2 + h^3\right) - 3\left(x^2 + 2xh + h^2\right) + 8x + 8h - 1$

Regrouping the terms, we have

$2x^3 + (6h-3)x^2 + \left(6h^2 - 6h + 8\right)x + \left(2h^3 - 3h^2 + 8h - 1\right)$. We require that $6h - 3 = 0 \Rightarrow h = \dfrac{1}{2}$.

For $h = \dfrac{1}{2}$, $B = 6h^2 - 6h + 8 = \dfrac{6}{4} - 3 + 8 = 6.5$, $C = 2h^3 - 3h^2 + 8h - 1 = \dfrac{1}{4} - \dfrac{3}{4} + 4 - 1 = -\dfrac{1}{2} + 3 = 2.5$

Thus, $(h, B, C) = \underline{\left(\mathbf{\dfrac{1}{2}, \dfrac{13}{2}, \dfrac{5}{2}}\right)}$ or $\underline{\mathbf{(0.5,\ 6.5,\ 2.5)}}$.

Alternate Solution (Norm Swanson – HW)

Based only on the inspection of the coefficients, the sum of the roots of $f(x) = 2x^3 - 3x^2 + 8x - 1$ is $\dfrac{3}{2}$. Thus, for the sum of the roots of the new cubic polynomial to be zero, each root will have to be reduced by $\dfrac{1}{3}$ of that amount, namely $h = \dfrac{1}{2}$. Then:

$(x - .5)^3 - 3(x - .5)^2 + 8(x - .5) - 1 = 2x^3 + Bx + C$ Applying synthetic division,

$$\begin{array}{l}
\quad\ \underline{2\ \ -3\ \ \ 8\ \ -1} \\
.5 \mid 2\ \ -2\ \ \ 7\ \ \ \mathbf{2.5} = C \\
.5 \mid 2\ \ -1\ \ \ \mathbf{6.5} = B \\
.5 \mid 2\ \ \ \mathbf{0} = \text{sum of roots} \\
.5 \mid \mathbf{2}\ = \text{lead coefficient}
\end{array}$$

Thus, the new polynomial is $2x^3 + 0x^2 + 6.5x + 2.5$ and the same result follows. You should think about why this shortcut works and avoids multiplying out the left-hand side.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2014 SOLUTION KEY

Round 2

A) The teen's <u>current</u> age, between 13 and 19 inclusive, can't be even because 2 years earlier or later he would not be"in his prime". We need test only <u>odd</u> cases.
$13 \Rightarrow$ 11 (prime), 15 (not prime) satisfies the stated conditions.
$15 \Rightarrow$ 13 (prime), 17 (prime) fails
$17 \Rightarrow$ 15 (not prime), 19 (prime) fails
$19 \Rightarrow$ 17 (prime), 21 (not prime)
Therefore, my current age is either 13 or 19 and I am no longer a teen in either 7 years or in 1 year. (M, m) = <u>**(7, 1)**</u>.

B) The least common multiple of 6 and 21 is 42.
We require that $42n > 2014 \Rightarrow n > 47^{+} \Rightarrow n_{\text{min}} = 48$
$48 \cdot 42 = \underline{\mathbf{2016}}$. Adding 42, we get additional possibilities: <u>**2058**</u> and <u>**2100**</u>.

Or, alternately, dividing 2014 by 42 result in a quotient of 47 and a remainder of 40
By adding 2 to the dividend (numerator), we insure divisibility by 42 and the same result follows.

C) The rightmost digit of an odd prime with 2 or more digits is 1, 3, 7, or 9.
Squaring these, the rightmost digit must be 1 or 9.
Therefore, we have eliminated two of the 5 given numbers.
3027 and 9025 are not the squares of a prime. The other 4 numbers <u>must</u> be.
$4489 = \left(6\boxed{x}\right)^2$, where x must be 3 or 7$\Rightarrow$ 67
$5329 = \left(7\boxed{x}\right)^2$, where x must be 3 or 7$\Rightarrow$ 73
$7921 = \left(8\boxed{x}\right)^2$, where x must be 1 or 9$\Rightarrow$ 89
Grouping them as follows decreases the probability of botching the arithmetic,
(67 +73) + 89 = 140 + 89 = <u>**229**</u>.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2014 SOLUTION KEY

Round 3

A) $\cos\left(Sin^{-1}(-.5)\right)+\cot\left(Tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right)+Cos^{-1}\left(\cos\frac{\pi}{6}\right)$

Let $A=Sin^{-1}(-0.5)$, $B=Tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$ and $C=Cos^{-1}\left(\cos\frac{\pi}{6}\right)$. Then:

$\sin A=-0.5$ and $-\frac{\pi}{2}<A<0\ (Q4)\ \Rightarrow\ A=-\frac{\pi}{6}$

$\tan B=-\frac{\sqrt{3}}{3}$ and $-\frac{\pi}{2}<B<0\ (Q4)\ \Rightarrow\ B=-\frac{\pi}{6}$

Cos^{-1} and $\cos$ are inverse functions **and** $\frac{\pi}{6}$ **is in the domain of** $Cos^{-1}\ \Rightarrow\ C=\frac{\pi}{6}$

Thus, $\cos\left(Sin^{-1}(-.5)\right)+\cot\left(Tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right)+Cos^{-1}\left(\cos\frac{\pi}{6}\right)=\cos\left(-\frac{\pi}{6}\right)+\cot\left(-\frac{\pi}{6}\right)+\frac{\pi}{6}$

$=\cos\left(-\frac{\pi}{6}\right)+\cot\left(-\frac{\pi}{6}\right)+\frac{\pi}{6}=\frac{\sqrt{3}}{2}-\sqrt{3}+\frac{\pi}{6}=\frac{\pi-3\sqrt{3}}{6}\Rightarrow(A,B)=\underline{\mathbf{\left(3\sqrt{3},6\right)}}$.

B) $\cos 290°=\cos(360°-70°)=\cos(-70°)=\cos(70°)$

Expanding using $\cos(A+B)=\cos A\cos B-\sin A\sin B$,

$\cos(x+150°)+\cos(x+30°)=(\cos x\cos 150°-\sin x\sin 150°)+(\cos x\cos 30°-\sin x\sin 30°)$

$=\left(\cos x\cdot-\frac{\sqrt{3}}{2}-\sin x\cdot\frac{1}{2}\right)+\left(\cos x\cdot\frac{\sqrt{3}}{2}-\sin x\cdot\frac{1}{2}\right)=-\sin x$

Thus, we have $\sin x=-\cos 70°=-\sin 20°$.

We require related angles in quadrants 3 and 4, $\begin{cases}180°+20°\Rightarrow \underline{\mathbf{200°}}\\ 360°-20°\Rightarrow \underline{\mathbf{340°}}\end{cases}$.

C) Let $B=Sin^{-1}\left(-\frac{4}{5}\right)$.

$\sin\left(2B+\frac{\pi}{2}\right)=\sin 2B\cos\left(\frac{\pi}{2}\right)+\cos 2B\sin\left(\frac{\pi}{2}\right)=(\sin 2B)(0)+(\cos 2B)(1)=\cos 2B$

$=1-2\sin^2 B=1-2\left(-\frac{4}{5}\right)^2=\underline{\mathbf{-\frac{7}{25}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2014 SOLUTION KEY

Round 4

A) Since Bonnie works three times as fast as Clyde, Clyde does $\frac{1}{4}$ of the job and Bonnie does $\frac{3}{4}$.

Since Bonnie take 6 minutes to do $\frac{3}{4}$ of the job, it will take her $\frac{4}{3}(6)=\underline{\mathbf{8}}$ minutes to do the entire job alone.

B) Let *x* denote my teenage son's age <u>and</u> shoe size back in the day.
Let *y* denote his <u>current</u> age. Then:

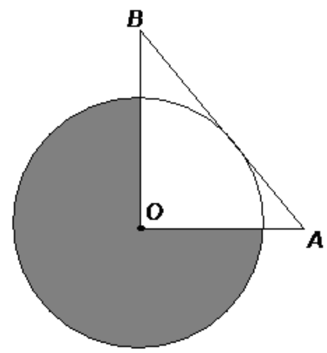
$$\begin{cases} y = x + 26 \\ x + 1 = \frac{3}{8}y \end{cases} \Rightarrow 8(x+1) = 3(x+26) \Rightarrow 5x = 78 - 8 \Rightarrow x = 14$$

Thus, his current shoe size is <u>**15**</u>.

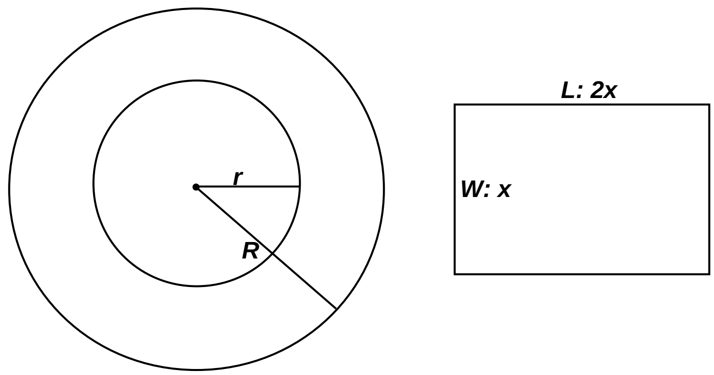
C) Let *A*, *B*, *C* and *D* denote the unknown round scores.
Since there are 7 rounds, the median score is the 4^{th} score when the scores are listed in increasing order: *A*, 18, 18, 22, *B*, *C*, *D*
The team score is <u>58</u> points plus the scores in the remaining 4 rounds.
A must be the team round score and must be less than 18, since the mode is 18 and it only occurred twice.
B, *C* and *D* must all be different (otherwise the mode would not be unique).
Also each must be larger than 22.
In the worst case scenario, (*A*, *B*, *C*, *D*) = (0, 24, 26, 28) $\Rightarrow$
total score = 58 + 3(26) = 136
In the best case scenario, (*A*, *B*, *C*, *D*) = (15, 26, 28, 30)
$\Rightarrow$ total score = 15 + 58 + 3(28) = 157
Thus, (*M*, *m*) = <u>**(157, 136)**</u>.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2014 SOLUTION KEY

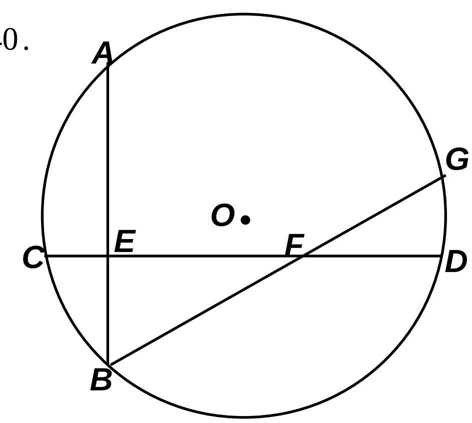
Round 5



A) The radius of circle O is the altitude to the hypotenuse.
$\frac{1}{2}\cdot 3\cdot 4=\frac{1}{2}\cdot 5\cdot h \Rightarrow h=\frac{12}{5}$
The shaded region is $\frac{3}{4}$ of the circle. $\frac{3}{4}\cdot\pi\cdot\left(\frac{12}{5}\right)^2=\underline{\boldsymbol{\frac{108\pi}{25}}}$.



B) Let x denote the width of the rectangle,
r the radius of the smaller circle and
R the radius of the larger circle. We are given that $D_{small}=2r=\frac{2}{3}R$ and $\mathrm{A}(\text{ring})=A(\text{rect})$.
$\Rightarrow R=3r$ and $8\pi r^2=2x^2 \Rightarrow x=2\sqrt{\pi}r$
Thus, $\frac{L}{D_{large}}=\frac{2x}{6r}=\frac{x}{3r}=\frac{2\sqrt{\pi}r}{3r}=\underline{\boldsymbol{\frac{2\sqrt{\pi}}{3}}}$.



C) According to the product chord theorem, $AE\cdot BE=2\cdot 20=40$.
Since AE and BE are integers and $AE>BE$, the possible ordered pairs (AE,BE) are: $(8,5),(10,4),(20,2),(40,1)$
The shortest possible length of $\overline{AB}$ is $8+5=13$ and $(BE,BF)=(5,13)\Rightarrow EF=12$.
Using F as a division point and re-applying the Product-Chord theorem, $CF\cdot FD=BF\cdot FG \Rightarrow$
$(2+12)(22-14)=13\cdot FG \Rightarrow FG=\frac{14\cdot 8}{13}=\underline{\boldsymbol{\frac{112}{13}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2014 SOLUTION KEY

Round 6

A) By trial and error, starting with $\left(A,B,C\right)=\left(9,17,25\right)$,

$x=1 \Rightarrow \left(9,16,25\right)$ is not a geometric sequence.

$\frac{16}{9}\neq\frac{25}{16}$, since, cross multiplying, $16\cdot 16=256\neq 9\cdot 25=225$

$x=2 \Rightarrow \left(9,15,25\right)$ is a geometric sequence, since $R=\frac{15}{9}=\frac{25}{15}=\underline{\mathbf{\frac{5}{3}}}$.

Algebraic solution

$\frac{17-x}{9}=\frac{25}{17-x}\Leftrightarrow 289-34x+x^2=225\Leftrightarrow x^2-34x+64=0$

$\Rightarrow (x-2)(x-32)=0\Rightarrow x=2,32$

The second solution is rejected, since $B=-32<0$ and $R=+\frac{5}{3}$ only.

B) The new sum is $\frac{2n(2n+1)(2n+2)}{6}$. The old sum was $\frac{n(n+1)(n+2)}{6}$.

New = 7(Old) $\Rightarrow$ $2n(2n+1)(2n+2)=7\left(n(n+1)(n+2)\right)$

$\Rightarrow 4n(2n+1)\cancel{(n+1)}=7\left(n\cancel{(n+1)}(n+2)\right)\Rightarrow 8n^2+4n=7n^2+14n\Rightarrow n^2=10n$

$n\neq 0\Rightarrow n=\underline{\mathbf{10}}$

C) Let m denote the mean and M the median. We require that $m=M$.

The mean of the 5 numbers is $m=\frac{40+A}{5}$.

This expression can only be an integer if A is a multiple of 5.

Sorting the known values, we have 4, 7, 13, 16 and we consider possible multiples of 5.

$A<\underline{\mathbf{-5}}\Rightarrow m<7$ and M = 7 (rejected)

$A=-5\Rightarrow m=7$ and M = 7 Bingo!

$A=0\Rightarrow m=8$ and M = 7 (rejected)

$A=5\Rightarrow m=9$ and M = 7 (rejected)

$A=\underline{\mathbf{10}}\Rightarrow m=10$ and M = 10 Bingo!

$A=15\Rightarrow m=11$ and M = 13 (rejected)

$A=20\Rightarrow m=12$ and M = 13 (rejected)

$A=\underline{\mathbf{25}}\Rightarrow m=13$ and M = 13 Bingo!

$A>25\Rightarrow m>13$ and M = 13 (rejected)

Notice that, regardless of the value of A, 4 and 16 cannot be the median value.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2014 SOLUTION KEY

Team Round

A) Since $f(x)$ is odd, $f(-x) = -f(x)$ and $f(x) = Ax^5 + Bx^3 + Cx$.

$f(1) = 0 \Rightarrow$ (1) $A + B + C = 0$ (4) $C = -(A+B)$

$f(2) = 42 \Rightarrow$ (2) $32A + 8B + 2C = 42 \Rightarrow$ (5) $16A + 4B + C = 21$

$f\left(\frac{1}{2}\right) = \frac{3}{16} \Rightarrow$ (3) $\frac{A}{32} + \frac{B}{8} + \frac{C}{2} = \frac{3}{16}$ (6) $A + 4B + 16C = 6$

Substituting for C in (5), $15A + 3B = 21 \Rightarrow B = 7 - 5A$.

Substituting for B in (4), $C = -(A + 7 - 5B) = 4A - 7$

Substituting for B and C in (6), $A + 28 - 20A + 64A - 112 = 6 \Rightarrow 45A = 118 - 28 = 90 \Rightarrow A = 2$

Thus, $(A,B,C) = (2,-3,1) \Rightarrow f(x) = 2x^5 - 3x^3 + x$

$$f(x) = 2x^5 - 3x^3 + x = x(2x^4 - 3x^2 + 1) = x(2x^2 - 1)(x^2 - 1) = 0 \Rightarrow x = \underline{\mathbf{0,\ \pm 1,\ \pm \frac{\sqrt{2}}{2}}} .$$

B) Suppose the 5 weights were *d* = 54, *a*, *b*, *c*, *e* = 62.
There would be 10 possible pairings, namely *ab*, *ac*, *ad*, *ae*, *bc*, *bd*, *be*, *cd*, *ce* and *de*.
Note each individual weight occurs exactly 4 times. The total weight of the 10 pairs is 1156
Thus, the total weight is 4(*a* + *b* + *c* + *d* + *e*) = 1156 ⇒ *a* + *b* + *c* + *d* + *e* = 289.
⇒ *a* + *b* + *c* + 116 = 289 ⇒ *a* + *b* + *c* = 173 and $a \geq 55$ and $c \leq 61$
a = 55 ⇒ *b* + *c* = 118 ⇒ (*b*, *c*) = ~~(56, 62)~~, (57, 61), (58, 60)
a = 56 ⇒ *b* + *c* = 117 ⇒ (*b*, *c*) = (57, 60), (58, 59)
a = 57 ⇒ *b* + *c* = 116 ⇒ (*b*, *c*) = ~~(58, 58)~~
a > 57 produces no additional ordered pairs.
Therefore, there are 4 <u>possible</u> ordered triples (*a*, *b*, *c*). However, 6 of the pair-sums are odd and this eliminates all but (*54*, **55**, **57, 61**, *62*). Thus, there is only <u>**1**</u> ordered triple (*a*, *b*, *c*).
Alternate solution: (Lexington HS)
Since 110 is the sum of the two smallest numbers, $a = 56$. Since 121 is the sum of the two largest numbers, $c = 59$. Thus, $4(54 + 56 + b + 59 + 62) = 1156 \Rightarrow b = 58$ and there can be only <u>**one**</u> quintuple.

C) $y = Sin^{-1}(6x^2 - 5x) \Rightarrow \sin(y) = 6x^2 - 5x \Rightarrow -1 \leq 6x^2 - 5x \leq 1$

$y = Sin^{-1}(6x^2 - 5x) \Rightarrow \sin(y) = 6x^2 - 5x \Rightarrow -1 \leq 6x^2 - 5x \leq 1$

$\Leftrightarrow 6x^2 - 5x \geq -1$ and $6x^2 - 5x \leq 1 \Leftrightarrow 6x^2 - 5x + 1 \geq 0$ and $6x^2 - 5x - 1 \leq 0$

$\Leftrightarrow (3x-1)(2x-1) \geq 0$ and $(6x+1)(x-1) \leq 0 \Leftrightarrow \left(x \leq \frac{1}{3} \text{ or } x \geq \frac{1}{2}\right)$ and $\left(-\frac{1}{6} \leq x \leq 1\right)$

Taking the intersection of these two rays and the overlapping segment, we have

$\underline{\mathbf{-\frac{1}{6} \leq x \leq \frac{1}{3} \text{ or } \frac{1}{2} \leq x \leq 1}}$ or using interval notation, $\underline{\left[-\frac{1}{6},\frac{1}{3}\right],\left[\frac{1}{2},1\right]}$. In each case, the word "or" , a comma or the union operator "U" may be used as the connector.

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D) Let the 5 notes be designated Q, H, W, D and E.

Then: $\begin{cases} Q+H+W+D+E=48 \\ Q+2H+4W+1.5D+.5E=16\cdot 4=64 \\ D=E=x \\ Q=7E=7x \\ Q=2H+4W \end{cases}$

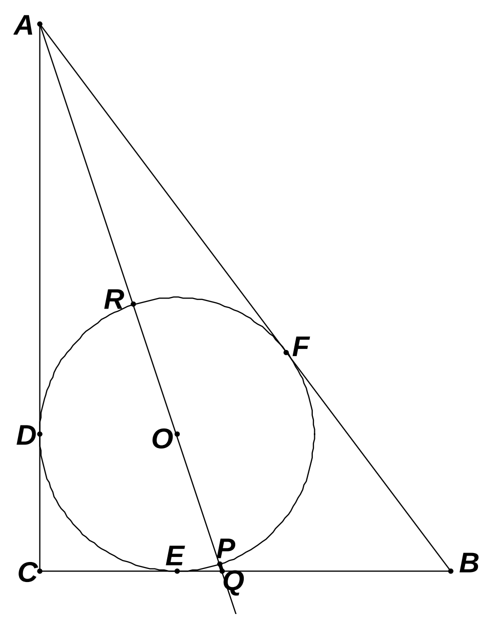
$Q+\left(2H+4W\right)+\left(1.5D+.5E\right)=64 \Leftrightarrow 2Q+2E=64 \Leftrightarrow 14x+2x=64 \Rightarrow x=4,\ Q=28$

$Q+H+W+D+E=48 \Leftrightarrow 28+H+W+2(4)=48 \Rightarrow \underline{H+W=12}$

$Q+\left(2H+4W\right)+\left(1.5D+.5E\right)=64 \Leftrightarrow 28+2H+4W+8=64 \Leftrightarrow \underline{H+2W=14}$

Subtracting, $W=2$ and $H=\underline{\mathbf{10}}$.

E) $\overrightarrow{AO}$ is an angle bisector $\Rightarrow \dfrac{QC}{4}=\dfrac{3-QC}{5} \Rightarrow QC=\dfrac{4}{3}$.



The radius of the inscribed circle is given by

$r=\dfrac{A}{s}=\dfrac{\frac{1}{2}\cdot 3\cdot 4}{\frac{1}{2}\left(3+4+5\right)}=\dfrac{6}{6}=1$. Since $OECD$ must be a square,

$EQ=\dfrac{1}{3} \Rightarrow OQ^2=1^2+\left(\dfrac{1}{3}\right)^2 \Rightarrow OQ=\dfrac{\sqrt{10}}{3}$. Applying the Pythagorean Theorem to ΔADO, $AO=\sqrt{10}$.

The required ratio $\dfrac{AO}{PQ}$ is

$\dfrac{\sqrt{10}}{\frac{\sqrt{10}}{3}-1}=\dfrac{3\sqrt{10}}{\sqrt{10}-3}\cdot\dfrac{\sqrt{10}+3}{\sqrt{10}+3}=\dfrac{30+9\sqrt{10}}{10-9}=\dfrac{30+9\sqrt{10}}{1}$

$\Rightarrow k=\underline{\mathbf{30+9\sqrt{10}}}$ or $\underline{\mathbf{3\left(10+3\sqrt{10}\right)}}$.

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F) $P_n(x) = xP_{n-1}(x) - P_{n-2}(x)$, where $P_0(x) = 1$ and $P_1(x) = 1 \Rightarrow$

$P_2(x) = xP_1(x) - P_0(x) = x(1) - 1 = x - 1$
$P_3(x) = xP_2(x) - P_1(x) = x(x-1) - 1 = x^2 - x - 1$
$P_4(x) = xP_3(x) - P_2(x) = x(x^2 - x - 1) - (x - 1) = x^3 - x^2 - 2x + 1$ etc.
$\sum_{n=0}^{n=6} P_n(3) = P_0(3) + P_1(3) + P_2(3) + P_3(3) + P_4(3) + P_5(3) + P_6(3)$

Rather than trying to evaluate explicit formulas for each of the polynomials P_i, let's use the recursive definition.
$P_2(3) = 3(1) - 1 = 2$
$P_3(3) = 3(2) - 1 = 5$
$P_4(3) = 3(5) - 2 = 13$
$P_5(3) = 3(13) - 5 = 34$
$P_6(3) = 3(34) - 13 = 89$
Thus, $\sum_{n=0}^{n=6} P_n(3) = 1 + 1 + 2 + 5 + 13 + 34 + 89 = \underline{\mathbf{145}}$.
Notice that from 2 on it appears that we are getting every other Fibonacci number.
Recall that the Fibonacci sequence was defined by
$a_n = a_{n-1} + a_{n-2}$, where $a_0 = a_1 = 1$.
Specifically, the sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …
What other interesting facts came be derived from this innocent sequence of polynomials?
Time will tell.
Send your ideas to olson.re@gmail.com.