MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2014
ROUND 1 COMPLEX NUMBERS (No Trig)

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Given: $(3+4i)^2 + a + bi = 3 + ki$, where a and b are real.
Compute k such that $a = -b$.

B) Given: $\sqrt{12-5i} = x + yi$ for real numbers x and y.
Compute $\dfrac{x^2}{y^4}$.

C) If $\dfrac{\left(3\sqrt{3}-3i\right)^3 - \left(3\sqrt{3}+3i\right)^3}{108} = x + yi$, compute $x^3 + y^3$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2014 SOLUTION KEY

Round 1

A) $(3+4i)^2 + a + bi = 9 - 16 + 24i + a + bi = (a-7) + (b+24)i = 3 + ki$
Equating the real and imaginary parts, $a = 10$ and $k = b + 24 \Leftrightarrow b = k - 24$.
Therefore, $k - 24 = -10 \Rightarrow k = \underline{\mathbf{14}}$.

B) Squaring both sides of $\sqrt{12-5i} = x + yi$, $\begin{cases} x^2 - y^2 = 12 \\ 2xy = -5 \\ x^2 + y^2 = 13 \end{cases}$.

Where did this third equation come from? We could have solved the second equation for *y* in terms of *x* and substituted in the first, but adding the first and third will be much easier.
Consider $|12-5i|$ - the absolute value of the radicand. As on the real number line, the absolute value of a complex number is its distance from the origin $O(0,0)$. Let $x + yi$ be represented by the point $P(x, y)$ in the complex plane and we have $|x+yi| = OP = \sqrt{x^2+y^2}$, regardless of the quadrant in which *P* is located.
Extracting, $\sqrt{(12)^2 + (-5)^2} = \sqrt{144+25} = \sqrt{169} = 13$ (or recall the Pythagorean Triple 5-12-13)
Thus, adding the first and third equations and dividing by 2, $x^2 = \dfrac{12+13}{2} = \dfrac{25}{2}$.
Subtracting the same equations, $2y^2 = 1 \Rightarrow y^4 = \dfrac{1}{4}$. Thus, $\dfrac{x^2}{y^4} = \underline{\mathbf{50}}$. Note that:
$\sqrt{12-5i}$ denotes either $\dfrac{5\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i$ or $-\dfrac{5\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i$. In both cases, $2xy = -5$ and $x^2 - y^2 = 12$.

C) Simply! Simplify! Simplify!
Recall: $(A \pm B)^3 = A^3 \pm 3A^2B + 3AB^2 \pm B^3$
Thus, in the expansion, the first and third terms cancel
Therefore, $(A+Bi)^3 - (A-Bi)^3 = 6A^2Bi - 2B^3i = 2B(3A^2 - B^2)i$.

$$\frac{\left(3\sqrt{3}-3i\right)^3 - \left(3\sqrt{3}+3i\right)^3}{108} = \frac{\cancel{3^3}\left(\left(\sqrt{3}-i\right)^3 - \left(\sqrt{3}+i\right)^3\right)}{\cancel{3^3}\cdot 4}$$

Since $A = \sqrt{3}$ and $B = -1$, we have $x + yi = \dfrac{2\cdot(-1)\cdot\left(3\cdot 3 - (-1)^2\right)i}{4} = -4i \Rightarrow (x,y) = (0,-4)$.
Thus, $x^3 + y^3 = (-4)^3 = \underline{\mathbf{-64}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2014
ROUND 2 ALGEBRA 1: ANYTHING

ANSWERS

A) x = ______________________________

B) ______________________________

C) (________ , _________ , ________)

A) For some positive constant A, if $x = 1$ is a solution of the equation $|x - A| = 5$, what is the other solution?

B) Given: $(2x - A)(3x + B) = 0$ for positive integer constants A and B.
Compute $A^2 - B^2$, if $A + B = 7$ and the sum of the solutions (for x) is an integer.

C) Given: $x \# y = \frac{x}{2} + \frac{y}{3}$ for integers x and y.
For some minimum integer $k > 10$, $\begin{cases} x \# y = k \\ 2x + 3y = k \end{cases}$. Compute the ordered triple (k, x, y).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2014 SOLUTION KEY

Round 2

A) $|1-A|=5 \Leftrightarrow 1-A=\pm 5 \Leftrightarrow A=1\pm 5=\cancel{-4}, 6$

$|x-6|=5 \Leftrightarrow x-6=\pm 5 \Leftrightarrow x=1, \underline{\mathbf{11}}$

B) The sum of the solutions is $\frac{A}{2}+\frac{-B}{3}=\frac{3A-2B}{6}$.

Testing the 6 possible ordered pairs (6, 1), (5, 2), (4, 3), (3, 4), (2, 5) and (1, 6),

only (4, 3) produces an integer solution sum $\left[\frac{3(4)-2(3)}{6}=1\right] \Rightarrow A^2-B^2=16-9=\underline{\mathbf{7}}$.

C) $\frac{x}{2}+\frac{y}{3}=k \Leftrightarrow$ (1) $3x+2y=6k$

(2) $2x+3y=k$

Adding the two equations and dividing by 5, we have $x+y=\frac{7k}{5}$.

Since we were given that $k>10$ and x and y must be integers, $k_{\min}=15$.

Substituting for y in (2), $2x+3(21-x)=15 \Rightarrow x=48 \Rightarrow \underline{(\mathbf{15, 48, -27})}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2014
ROUND 3 PLANE GEOMETRY: AREAS OF RECTILINEAR FIGURES

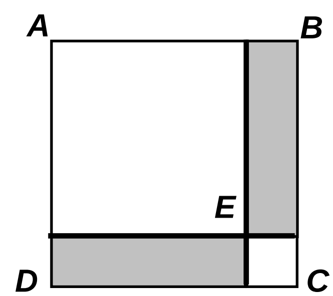
ANSWERS

A) ______________________________

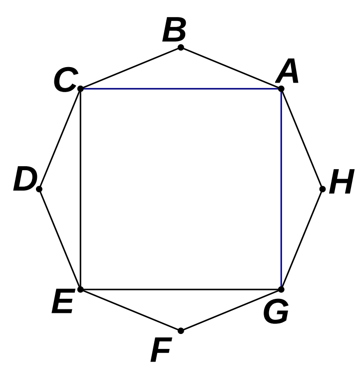
B) ______________________________

C) ______________________________

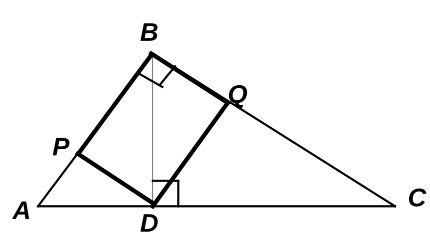
A) *ABCD*, a square with area 225, is subdivided into 2 squares and 2 rectangles by perpendiculars that intersect at point *E*. If $CE = \sqrt{32}$, compute the area of the shaded region.



B) A side of the regular octagon *ABCDEFGH* is $\sqrt{2}$. Compute the area of the square *ACEG*.



C) In ΔABC, the altitude is drawn to the hypotenuse of a 3 – 4 – 5 right triangle, intersecting the hypotenuse in point *D*. From point *D*, altitudes are drawn to the legs, intersecting $\overline{AB}$ in point *P* and intersecting $\overline{BC}$ in point *Q*. Compute the area of rectangle *DPBQ*, as a ratio of relatively prime integers.



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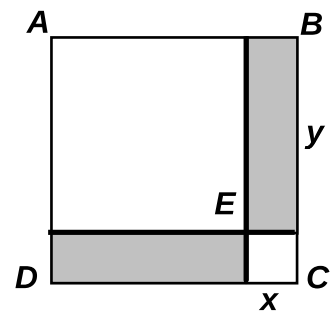
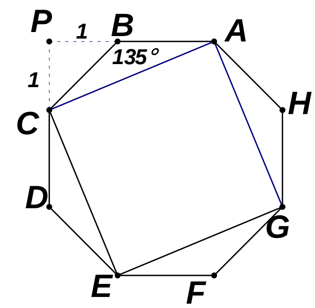


Round 3

A) $(x+y)^2 = 225 \Rightarrow AB = 15$

$CE = \sqrt{32} \Rightarrow x = 4, y = 11$

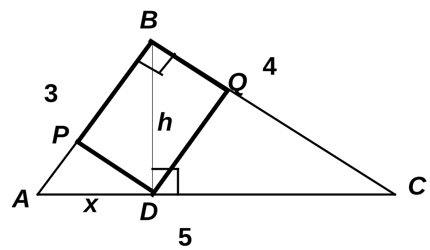
Therefore, the area of the shaded region is $2(4 \cdot 11) = \underline{\mathbf{88}}$.



B) Using Pythagorean Theorem on right ΔAPC,

$AC^2 = \left(1+\sqrt{2}\right)^2 + 1^2$

$\Rightarrow AC^2 = 1 + 2\sqrt{2} + 2 + 1 = \underline{\mathbf{4+2\sqrt{2}}}$



C) The area of ΔABC is $\frac{1}{2}\cdot 3\cdot 4 = \frac{1}{2}\cdot 5\cdot h \Rightarrow h = \frac{12}{5}$.

Proceed by the Pythagorean Theorem

$9 = x^2 + \left(\frac{12}{5}\right)^2 \Rightarrow x^2 = \frac{9\cdot 5^2 - 12^2}{5^2} = \frac{9(25-16)}{5^2} = \frac{9^2}{5^2}$

Thus, $AD = \frac{9}{5}$ and $CD = \frac{16}{5}$. Alternately,

$\Delta BAD \sim \Delta CAB \sim \Delta CBD \Rightarrow \frac{BA}{CB} = \frac{AD}{BD} \Rightarrow \frac{3}{4} = \frac{x}{h} \Rightarrow x = \left(\frac{3}{4}\right)\left(\frac{12}{5}\right) = \frac{9}{5}$ or, invoking the fact that the altitude to the hypotenuse is the mean proportional between the segments on the hypotenuse, $h^2 = x(5-x)$.

By similar arguments for triangles BAD and BCD, $DP = \frac{36}{25}$ and $DQ = \frac{48}{25} \Rightarrow$

$\frac{36}{25}\cdot\frac{48}{25} = \underline{\mathbf{\frac{1728}{625}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2014
ROUND 4 ALG 1: FACTORING AND ITS APPLICATIONS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) Compute the greatest common factor of $42x^2yz^3$ and $90x^3z^4$.

B) Compute <u>all</u> <u>rational</u> values of x satisfying $(3x+4)(8x-5)=-23$.

C) Given: $x(x-2A)+A(A+5)-4=5(x+4)$
Solve for x in terms of A.

MASSACHUSETTS MATHEMATICS LEAGUE
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Round 4

A) As the product of primes, 42 = 2(3)(7) and 90 = $2(3)^2 5$.
Taking the smallest exponents of the common factors,
The numerical component is 2(3) = 6 and the literal component is x^2z^3.
Therefore, the GCF is $\underline{\mathbf{6x^2z^3}}$.

B) Since the product is not equal to zero, the factorization does not help. Multiplying out the left side and combining like terms, we have
$(3x+4)(8x-5)=-23 \Leftrightarrow 24x^2+17x+3=0$. Since the coefficient of the middle term is odd, the factors of 24 cannot both be even, leaving only possibilities of $24\cdot 1$ or $8\cdot 3$. If these fail, we would have to use the quadratic formula and none of the solutions would be rational.
Since $24x^2+17x+3=(8x+3)(3x+1)=0$ and we have rational solutions, namely,
$x = \underline{\mathbf{-\frac{3}{8}, -\frac{1}{3}}}$.

C) $x(x-2A)+A(A+5)-4=5(x+4) \Leftrightarrow (x^2-2Ax+A^2)+5A-5x-24=0$
$\Leftrightarrow (x-A)^2-5(x-A)-24=0$
$\Leftrightarrow (x-A-8)(x-A+3)=0$
$\Rightarrow x=\underline{\mathbf{A+8}}$ or $x=\underline{\mathbf{A-3}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2014
ROUND 5 TRIG: FUNCTIONS OF SPECIAL ANGLES

ANSWERS

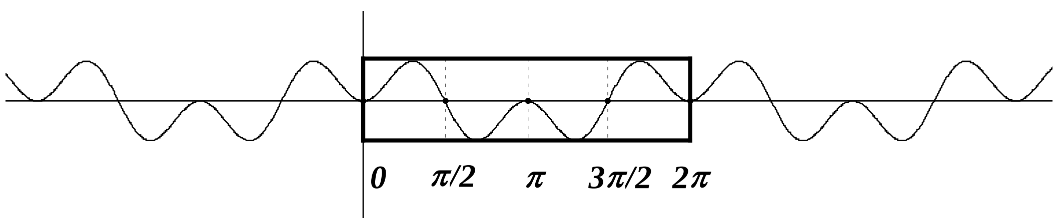
A) ______________________

B) ______________________

C) (____________ , ____________)

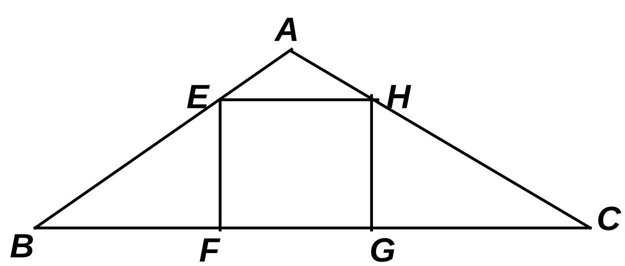
A) Determine <u>all</u> values of x for which $\dfrac{\tan x - 1}{(2\sin x - 1)\cos x(\tan^2 x - 3)}$ is <u>un</u>defined over the interval $0° \le x \le 180°$. Express your answer(s) in degrees.

B) The graph of $f(x) = \sin(x)\sin(2x)$ has a period of 2π as is easily seen in the graph below



Compute $f\left(\dfrac{-602\pi}{3}\right) + f\left(\dfrac{602\pi}{3}\right) - \left(f\left(\dfrac{\pi}{4}\right)\right)^2$.

C) Given: $m\angle A = 120°$, $AB = AC = 11$
$EFGH$ is a square
If the area of $EFGH$ is $M - N\sqrt{3}$,
compute the ordered pair (M, N).



MASSACHUSETTS MATHEMATICS LEAGUE
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Round 5

A) The value of the fraction is undefined only when the denominator is zero, or when one of the trig functions is undefined. We do not consider values of x for which the numerator is zero, namely 45°, since for this value the denominator is not 0.

$2\sin x - 1 = 0 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \underline{\mathbf{30°, 150°}}$

$\cos x = 0 \Rightarrow x = \underline{\mathbf{90°}}$ (Also, $\tan x$ is undefined for $x = 90°$.)

$\tan^2 x - 3 = 0 \Rightarrow \tan x = \pm\sqrt{3} \Rightarrow x = \underline{\mathbf{60°, 120°}}$

B) Since sin(–x) = –sin(x) and sin(–2x) = –sin(2x), it follows that
$f(-x) = \sin(-x)\sin(-2x) = \sin(x)\sin(2x) = f(x)$. Thus, since
$\frac{602\pi}{3} = 200\frac{2}{3}\pi$, with a period of 2π, we can disregard 200π.

$$f\left(\frac{-602\pi}{3}\right) = f\left(\frac{602\pi}{3}\right) = f\left(\frac{2\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{2}\cdot\frac{-\sqrt{3}}{2} = -\frac{3}{4}$$

$$f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)\cdot\sin\left(\frac{\pi}{2}\right) = \frac{\sqrt{2}}{2}\cdot 1 = \frac{\sqrt{2}}{2}$$

With these observations, the given expression evaluates to $2\left(-\frac{3}{4}\right) - \frac{1}{2} = \underline{\mathbf{-2}}$.

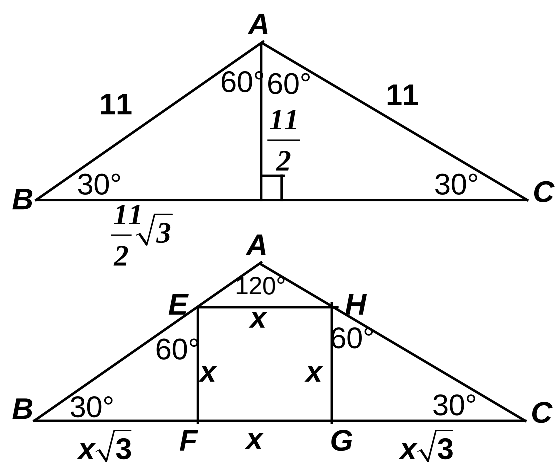
(Ask your coach/teammates about even and odd functions.)

C) $BF + FG + GC = BC \Rightarrow 2x\sqrt{3} + x = 11\sqrt{3} \Rightarrow x = \frac{11\sqrt{3}}{2\sqrt{3}+1}\cdot\frac{2\sqrt{3}-1}{2\sqrt{3}-1} = \frac{66 - 11\sqrt{3}}{11} = 6 - \sqrt{3}$

Therefore, the area of $EFGH$ is

$\left(6-\sqrt{3}\right)^2 = 36 - 12\sqrt{3} + 3 = 39 - 12\sqrt{3}$

$\Rightarrow (M, N) = \underline{(\mathbf{39,12})}$.



MASSACHUSETTS MATHEMATICS LEAGUE
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ROUND 6 PLANE GEOMETRY: ANGLES, TRIANGLES AND PARALLELS

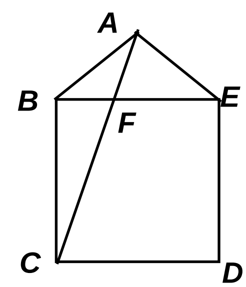
ANSWERS

A) _______________________________

B) _______________________________

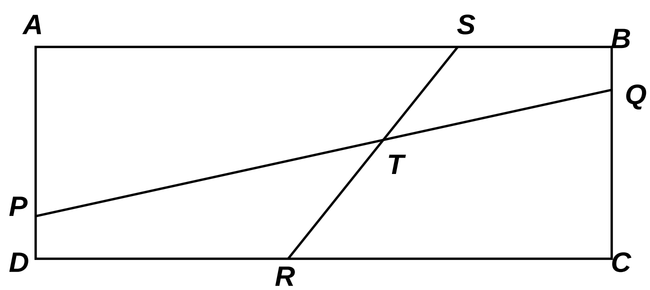
C) _______________________________

A) ABE is an isosceles triangle with base $\overline{BE}$.
$BCDE$ is a square, $m\measuredangle BAE = 114°$ and $\overrightarrow{AC}$ trisects $\measuredangle BAE$.
Compute $m\measuredangle ACD$.



B) In scalene triangle ABC, $m\angle A = (6x+7)°$, $m\angle B = (8x-9)°$ and the exterior angle at C has a measure of $(x^2+46)°$. Compute all possible values of x.

C) In rectangle $ABCD$,
$m\angle AST = (5x-11)°, m\angle PQC = (2x+15)°$,
where x is an integer.
Given that $\angle PTS$ is obtuse, compute the number of possible degree-measures of $\angle PTR$.



MASSACHUSETTS MATHEMATICS LEAGUE
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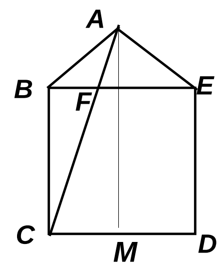
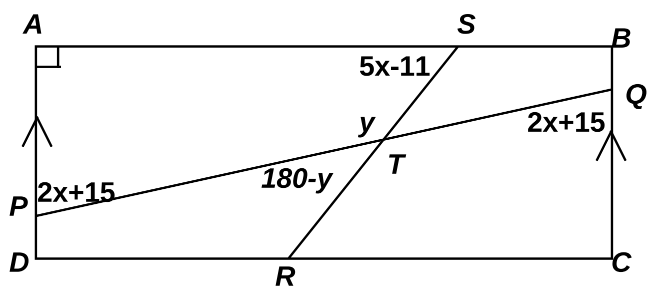
Round 6



A) Drop a perpendicular from A to $\overline{CD}$.
$m\measuredangle EAM = m\measuredangle BAM = 57°$.
$m\measuredangle BAF = \frac{1}{3}\cdot 114 = 38° \Rightarrow m\measuredangle CAM = 19°$
$\Rightarrow m\measuredangle ACD = m\measuredangle ACM = 90 - 19 = \underline{\mathbf{71}}°$

B) Since the measure of the exterior angle equals the sum of the measures of the two remote interior angles, we have $(6x+7)+(8x-9) = x^2+46 \Leftrightarrow x^2-14x+48 = (x-6)(x-8) = 0$.

For $x = 8$, $m\angle A = m\angle B = 55°$ and ABC is isosceles. This solution is rejected.
For $x = 6$, $m\angle A = 43$, $m\angle B = 39°$, and $m\angle C = 180-(43+39) = 98$ and ABC is scalene.
Thus, $x = \underline{\mathbf{6}}$ only.



C) In quadrilateral $PAST$,
$7x+4+90+y = 360 \Rightarrow y = 266-7x > 90$
$\Rightarrow 7x < 176 \Rightarrow x \le 25$
But we also know that $y < 180$
$\Rightarrow 7x > 86 \Rightarrow x \ge 13$
Thus, $13 \le x \le 25$ generates all possible values of $m\angle PTR$, a total of $\underline{\mathbf{13}}$ different values.
Check: For $x = 13,...,25$,
$m\angle PTR = 180 - y = 7x - 86° \Rightarrow 5°, 12°\ (5+7\cdot 1),...,89°\ \left(5+7\cdot 12\right)$ - 13 distinct values.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2014
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________________ D) ____________________________

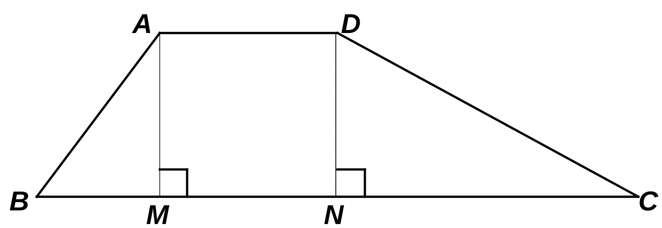
B) SUN MON TUE WED THU FRI SAT E) ____________________________

C) ____________________________ F) ____________________________

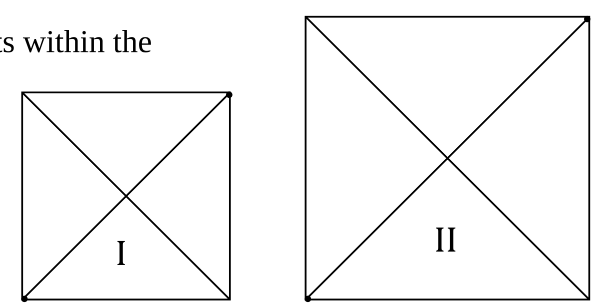
A) Let $N = \dfrac{1}{(1-i)^k}$ for integer values of k.
If $10 < k < 100$, determine for <u>how many</u> values of k, N is real.

B) Misao Okawa, one of the oldest living persons in the world, celebrates his birthday in March. In 2014, his birthday fell on a Wednesday. On what day of the week did his birthday fall in 1898, the year he was born?
Recall that there are 365 days in a year, except in leap years. The extra day (2/29) is added only in non-century years divisible by 4 and in century years divisible by 400.

C) Given: Trapezoid $ABCD$ with $\overline{AD} \parallel \overline{BC}$ and $(AB, BC, CD, AM) = (30, 87, 51, 24)$
An <u>isosceles</u> trapezoid $PQRS$ has the same perimeter as $ABCD$, sides of integer length and an altitude equal in length to the altitude of $ABCD$. Compute <u>all</u> possible areas of trapezoid $PQRS$.



D) In each of the squares below, consider the lattice points within the triangular regions marked I and II. The lower left vertex in each square is the origin. The upper right vertices are (n, n) and $(n + 1, n + 1)$ respectively, where n is a positive integer. Compute all value of n for which the number of lattice points in region II is 5 more than the number of lattice points in region I.



E) Compute <u>all</u> possible values of $\sin\left(\dfrac{n\pi}{3} + \dfrac{m\pi}{6}\right)$, if m and n are both positive multiples of 3.

F) ΔABC is scalene and acute.
Its interior angles measure $x°$, $y°$ and $(3x - 2y)°$, where x and y are integers.
If $x + y < 120$, compute the <u>number</u> of possible values of x.

MASSACHUSETTS MATHEMATICS LEAGUE
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Team Round

A) Note that $N = \dfrac{1}{(1-i)^k} = \left(\dfrac{1}{1-i}\right)^k = \left(\dfrac{1+i}{2}\right)^k = 2^{-k}\cdot(1+i)^k$

Since 2^{-k} is real for all integer values of *k* in the specified range, *N* is real whenever $(1+i)^k$ is real.

For $k = 1$ to 4, $(1+i)^k = 1 + i, 2i, -2 + 2i, -4$.

For $k = 5$ to 8, the values obtained are obtained by multiplying the above values by -4.
There is one real value of *N* in every block of four consecutive *k*-values.
If *k* is a multiple of 4, *N* is real. *N* is real for $k = 12, 16, \ldots, 96$, or $4(3, 4, \ldots, 24)$.
Thus, we have a total of **22** different values of *k*.

B) 7 days before (or after) a given date will fall on the same day of the week.
Since $365 = 7\cdot 52 + 1$, a non-leap year consists of 52 weeks and 1 day.
Therefore, from year to year, a given date advances 1 day of the week (DOW), unless 2/29 falls between the two dates. Moving back in time, starting in 2014, consider the following sequence:

14	13	12	11	10	9	8	7	6	5	4
	-1	-1	-2	-1	-1	-1	-2	-1	-1	-1
WED	TUE	MON	SAT	FRI	THU	WED	MON	SUN	SAT	FRI

Since 2/29 falls between 3/2011 and 3/2012 and between 3/2007 and 3/2008, the day of the week changes by 2 days. Over a 4 year period, DOW changes by 5 days, unless the 4 year period spans a century year which is not divisible by 400 (like the year 1900). Therefore, 112 years ago (28 4-year periods), in 1902, the DOW has cycled through $28\cdot 5 = 140$ days. Since 140 is divisible by 7, in 1902, his birthday fell on the same DOW, namely Wednesday. Since none of the remaining 4 intervening years were leap years, the DOW changes only by 4 days. In 1898, his birthday fell on a **SAT**. You are invited to apply Zeller's formula to his actual birthdate 3/5/1898 to confirm this result.

$$z = \left[\frac{13m-1}{5}\right] + \left[\frac{y}{4}\right] + \left[\frac{c}{4}\right] + d + y - 2c \text{, where}$$

d denotes the day (1..31)
m denotes the month according to the following funky rule:
1 = March 2 = April … 10 = December and January and February are assigned 11 and 12 respectively **for the previous year**
c denotes the "century" in which the date falls (***YY****YY*)
y denotes the year (*YY****YY***), i.e. 0 ... 99
Now, divide *z* by 7.

The integer remainder determines the day of the week.
(0, Sunday), (1, Monday), (2, Tuesday)… (6, Saturday)

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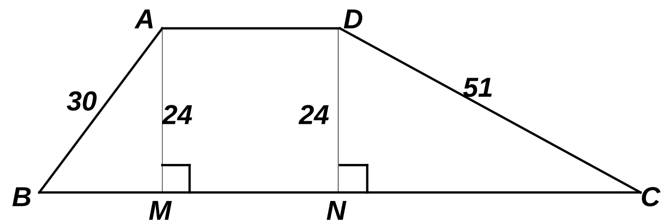
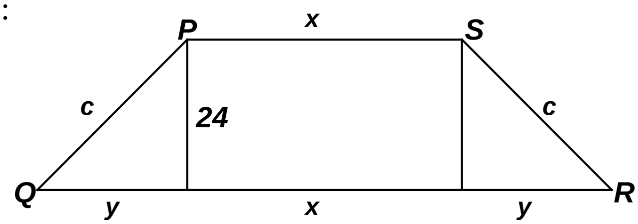


C) Drop perpendiculars from A and D to $\overline{BC}$.
$BM = 18$ (18, 24, 30) = 6(3, 4, 5)
$NC = 45$ (24, 45, 51) = 3(8, 15, 17)
$BC = 87 \Rightarrow MN = AD = (87 - 18 + 45) = 24$
$\Rightarrow$ Per($ABCD$) = 192
Per($PQRS$) = $2x + 2y + 2c = 192 \Rightarrow x + y + c = 96$ and (y, 24, c) is a Pythagorean triple
The possible triples (with a leg of 24) are:



1) 6(3, 4, 5) $\Rightarrow$ (18, 24, 30)
2) 8(3, 4, 5) $\Rightarrow$ (24, 32, 40)
3) 2(5, 12, 13) $\Rightarrow$ (10, 24, 26)
4) (7, 24, 25)
5) 3(8, 15, 17) $\Rightarrow$ (24, 45, 51)
6) (24, 143, 145)

Aside:

The system $\begin{cases} a = m^2 - n^2 \\ b = 2mn \\ c = m^2 + n^2 \end{cases}$ may be used to generate any Pythagorean triple. The triple will be primitive whenever the greatest common factor of m and n is 1.
The primitive triples 1) through 6) above were generated by (m, n) = (2, 1), (3, 2), (4, 3), (4,1) and (12, 1).
From the list above, the possible values of $y + c$ are:
48 for (y, c) = (18, 30), 72 for (y, c) = (32, 40), 36 for (y, c) = (10, 26), 32 for (y, c) = (7, 25),
~~96~~ for (y, c) = (45, 51) – rejected ($\Rightarrow x = 0$) ~~288~~ for (y, c) = (143, 145) – also rejected
$x = 96 - (y + c) \Rightarrow$ the allowable values of x are: 48, 24, 60, 64
Thus, the allowable corresponding values of (x, y) are: (48, 18), (24, 32), (60, 10) and (64, 7)

Since the area($PQRS$) = $24(x + y)$, we have $24 \cdot \begin{cases} 66 \\ 56 \\ 70 \\ 71 \end{cases} \Rightarrow$ **<u>1584, 1344, 1680, 1704</u>**

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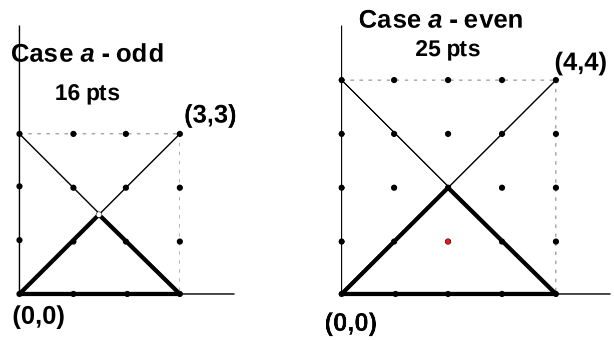
D) If a is odd, the diagonals do not intersect at a lattice point; for a even, they do.
Continuing to draw pics would quickly become tedious. Let's generalize.



Case even ($a = 4$):
There are $(a + 1)^2$ lattice points, but we must exclude:

points on the boundary: $4(a + 1) - 4 = 4a$
(4 · points on each edge – 4 corners which have been counted twice)
points on the diagonals (not already counted): $2\left(\left(a+1\right)-2\right)-1 = 2a-3$
(minus center point which is on both diagonals and has been counted twice)

Simplifying, $\left(a+1\right)^2 - 4a - (2a-3) = a^2 - 4a + 4 = (a-2)^2$

Since each of the 4 regions has the same number of lattice points, we have $\boxed{\dfrac{(a-2)^2}{4}}$ (a even)

Case odd ($a = 3$):
There are $(a + 1)^2$ lattice points, but we must exclude:
points on the boundary: $4a$
points on the diagonals: $2(a + 1) - 4 = 2(a - 1)$

Simplifying, $\left(a+1\right)^2 - 4a - 2(a-1) = a^2 - 4a + 3 = (a-1)(a-3) \Rightarrow \boxed{\dfrac{(a-1)(a-3)}{4}}$ (a odd)

n odd $\Rightarrow$ use $(n + 1)$ for a in the even formula:

$$\frac{\left((n+1)-2\right)^2}{4} - \frac{(n-1)(n-3)}{4} = 5 \Leftrightarrow \left(n^2-2n+1\right)-\left(n^2-4n+3\right) = 20 \Rightarrow 2n = 22 \Rightarrow n = \underline{\mathbf{11}}$$

n even $\Rightarrow$ use $(n + 1)$ for a in the odd formula:

$$\frac{\left((n+1)-1\right)\left((n+1)-3\right)}{4} - \frac{(n-2)^2}{4} = 5 \Leftrightarrow \left(n^2-2n\right)-\left(n^2-4n+4\right) = 20 \Rightarrow 2n = 24 \Rightarrow n = \underline{\mathbf{12}}$$

Ask you teammates/coach about Pic's Theorem.

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E) Let $m = 3j$ and $n = 3k$ for positive integers *j* and *k*.

$\frac{n\pi}{3} + \frac{m\pi}{6} = \frac{\pi}{6}(2n+m) = \frac{\pi}{6}(6j+3k) = \frac{\pi}{2}(2j+k)$

Since *j* and *k* are positive integers, so is $2j+k$.

Can we get all non-coterminal multiples of $\frac{\pi}{2}$?

Yes! $2j+k$ produces all and only quadrantal values

$(j,k) = (1,1) \Rightarrow \frac{3\pi}{2}$ and $\sin\left(\frac{3\pi}{2}\right) = \underline{\mathbf{-1}}$

$(j,k) = (1,2) \Rightarrow \frac{4\pi}{2}$ and $\sin(2\pi) = \underline{\mathbf{0}}$

$(j,k) = (2,1) \Rightarrow \frac{5\pi}{2}$ and $\sin\left(\frac{5\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = \underline{\mathbf{+1}}$

$(j,k) = (2,2) \Rightarrow \frac{6\pi}{2}$ and $\sin(3\pi) = \sin(\pi) = 0$

F) Let (A,B,C) denote the interior angles with measures $(x, y, 3x-2y)$.

Triangle Sum $\Rightarrow$ $4x - y = 180 \Rightarrow y = 4x - 180$

Substituting in $x + y < 120$, $5x < 300 \Rightarrow x < 60$.

Thus, our starting point is $x = 59 \Rightarrow (59, 56, 180 - 115 = 65)$

Decreasing *x* by 1, decreases *y* by 4, and consequently the third interior angle will increase by 5.
We must stop when the largest angle *C* becomes a right angle.
Possible measures of *C* are 65, 70, 75, 80 and 85, implying we have **<u>5</u>** possible *x*-values.

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Round 1 Algebra 2: Complex Numbers (No Trig)

A) 14   B) 50   C) –64

Round 2 Algebra 1: Anything

A) 11   B) 7   C) (15, 48,–27)

Round 3 Plane Geometry: Area of Rectilinear Figures

A) 88   B) $4+2\sqrt{2}$ C) $\frac{1728}{625}$

Round 4 Algebra: Factoring and its Applications

A) $6x^2z^3$ B) $-\frac{3}{8}, -\frac{1}{3}$ C) $x = A+8,\ A-3$

Round 5 Trig: Functions of Special Angles

A) 30, 60, 90, 120, 150   B) –2   C) (39, 12)
(answers with degree symbols are acceptable)

Round 6 Plane Geometry: Angles, Triangles and Parallels

A) 71   B) 6 only
[8 is extraneous.]   C) 13

Team Round

A) 22   D) 11 or 12

B) SAT   E) 0, ±1

C) 1344, 1584, 1680, 1704
(in any order)   F) 5