MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 – DECEMBER 2014
ROUND 1 TRIG: RIGHT ANGLE PROBLEMS, LAWS OF SINES AND COSINES

ANSWERS

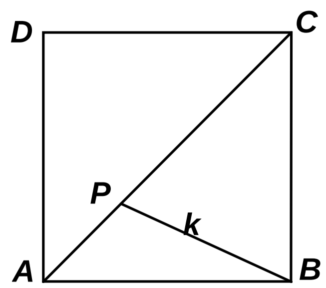
A) ______________________

B) (__________ , __________)

C) ______________________

A) The hypotenuse of right triangle ABC has a length of 195 units.
If all sides of ΔABC have integer lengths, compute the perimeters of <u>all</u> possible triangles ABC.

B) Point P is located on the diagonal $\overline{AC}$ (not at an endpoint) in square $ABCD$ whose side has length 6. $BP = k$. For the <u>minimum</u> integer value of k, $\sin\angle APB = q$.
Compute the ordered pair (k, q).



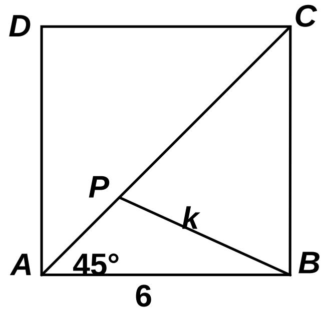
C) In ΔABC, $AC = 7$, $AB = 13$ and $\measuredangle A$ is <u>obtuse</u>.
If $\overline{BC}$ has integer length, what is the <u>maximum</u> value of $\cos A$?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014 SOLUTION KEY

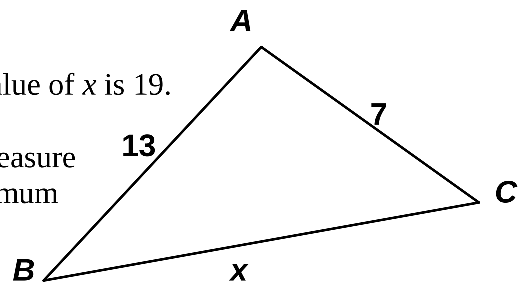
Round 1

A) Since $195 = 3 \cdot 5 \cdot 13$, the only integral right triangles with a hypotenuse of length 195 are triangles similar to a 3-4-5 or 5-12-13 right triangle, namely, triangles with sides of length $39(3,4,5) = (117,156,195) \Rightarrow$ **468** or $15(5,12,13) = (75,180,195) \Rightarrow$ **450**.

B) $AC = 6\sqrt{2}$. Since P cannot coincide with A, $k < 6$. When $\overline{BP} \perp \overline{AC}$, $k = 3\sqrt{2}$.
Thus, $3\sqrt{2} < BP < 6$ and the only integer value in this range is 5.
Using the Law of Sines on ΔAPB, $\dfrac{\sin 45°}{k} = \dfrac{\sin \angle APB}{6}$.
Substituting, $\sin \angle APB = \dfrac{3\sqrt{2}}{5} \Rightarrow \left(\mathbf{5}, \dfrac{\mathbf{3\sqrt{2}}}{\mathbf{5}}\right)$.



C) Let x denote the length of $\overline{BC}$.
Clearly, to satisfy the triangle inequality, the maximum value of x is 19.
As the value of x increases, so does the measure of $\measuredangle A$.
However, since the value of the cosine decreases as the measure of the angle increases from 90° to 180°, we want the minimum possible value of x for which $\measuredangle A$ is obtuse!
Using the Law of Cosines, we have



$$x^2 = 7^2 + 13^2 - 2 \cdot 7 \cdot 13 \cdot \cos A \Rightarrow \cos A = \frac{218 - x^2}{2 \cdot 7 \cdot 13}.$$

If $\measuredangle A$ is obtuse, then $\cos A < 0$ and this occurs when $218 - x^2 < 0 \Leftrightarrow x^2 > 218 \Rightarrow x > 14$.
Thus, the minimum value of x is 15 and $\cos A = \dfrac{218 - 15^2}{2 \cdot 7 \cdot 13} = \dfrac{-7}{2 \cdot 7 \cdot 13} = -\dfrac{\mathbf{1}}{\mathbf{26}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014
ROUND 2 ARITHMETIC/NUMBER THEORY

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Compute the remainder when 6^{153} is divided by 7.

B) Find the smallest of four consecutive positive integers such that the first (smallest) is divisible by 5, the second by 4, the third by 3, and the fourth by 2.

C) What is the second smallest positive integer with exactly 15 positive factors?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014 SOLUTION KEY

Round 2

A) Examine the remainders of consecutive powers of 6 divided by 7.

$6^1 = 6, 6^2 = 36, 6^3 = 216, 6^4 = 1296$

Dividing by 7, the corresponding remainders are 6, 1, 6, 1, … .
Thus, this pattern suggests that 6 raised to an odd power leaves a remainder of 6; an even power leaves 1. Therefore, 6^{153} divided by 7 leaves a remainder of **<u>6</u>**.

Alternate solution/proof (Modular arithmetic):
Note: The notation $a \equiv b \pmod{n}$ reads "*a* is congruent to *b* modulo *n*".
It is equivalent to saying "when *a* is divided by *n*, the remainder is *b*".
Convince yourself that this relation satisfies the transitive property, namely,
if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.

$6 \equiv -1 (\text{mod } 7) \Rightarrow 6^{153} \equiv (-1)^{153} = -1 (\text{mod } 7)$

By transitivity, $6^{153} \equiv \underline{\mathbf{6}} (\text{mod } 7)$.

B) The numbers –5, –4, –3, and –2 work, but are not positive. The LCM of 5, 4, 3, and 2 is 60. Adding 60 to each of these numbers, we have the desired 4 consecutive positive integers. The smallest is 60 + (-5) = **<u>55</u>**.
55 = 5(11), 56 = 4(14), 57 = 3(19) and 58 = 2(29)

Alternate solution #1 (Algebraic tour de force):
Let the integers be *x*, *x* + 1, *x* + 2 and *x* + 3. Increase the first by 5, the second by 4, the third by 3 and the fourth by 2 to get *x* + 5, *x* + 5, *x* + 5 and *x* + 5.
Thus, *x* + 5 must be divisible by 5, 4, 3, and 2, that is, by 60.
The smallest such positive integer is 60 – 5 = **<u>55</u>**.

Alternate Solution #2 (Brute Force – the first number must be a multiple of 5):
5, <u>*6*</u>, 7, 8 - fails 10, <u>*11*</u>, 12, 13 - fails 15, 16, <u>*17*</u>, 18 – fails … 35, 36, <u>37</u>, 38 – fails
… 55, 56, 57, 58 – Bingo!

C) A number with exactly 15 factors must be of the form p^{14} or p^4q^2, where *p* and *q* are primes. The smallest two numbers of the first form are $2^{14} = 2^{10} \cdot 2^4 = 16384$ and 3^{14} which is considerably larger. The smallest numbers of the second form are
$2^4 3^2 = 144,\ 2^4 5^2 = 400,\ 3^4 2^2 = 324$ and the winner is **<u>324</u>**.

Alternately, in order to have an odd number of factors, an integer must be a perfect square.
$4 = 2^2 \Rightarrow$ 3 factors $9 = 3^2 \Rightarrow$ 3 factors $16 = 2^4 \Rightarrow$ 5 factors $25 = 5^2 \Rightarrow$ 3 factors
$36 = 2^2 3^2 \Rightarrow (2+1)(2+1) = 9$ factors
… $144 = 2^4 3^2 \Rightarrow (4+1)(2+1) = 15$ factors (1st)
… $\underline{\mathbf{324}} = 18^2 = 2^2 3^4 \Rightarrow (2+1)(4+1) = 15$ factors (2nd)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014
ROUND 3 COORDINATE GEOMETRY OF LINES AND CIRCLES

ANSWERS

A) (_____ , _____) r = __________

B) (______________ , ______________)

C) (______________ , ______________)

A) A student was in the middle of completing the square to determine the center and the radius of a circle when he was called to the dinner table. Complete this unfinished business, that is, give the coordinates of the center and the radius of this circle.

$$\left(x^2 - 16x + \_\_\_\right) + \left(y^2 + 10y + \_\_\_\right) = 11 + \_\_\_\_$$

B) Line L_1 passes through the point $A(-3, 1)$ and has slope -1.5 .
Line L_2 is the perpendicular bisector of the segment whose endpoints are $B(4, 3)$ and $C(0, 7)$.
$P(x, y) = L_1 \cap L_2$. Compute the ordered pair (x, y).
Note: $\cap$ signifies "the intersection of ".

C) Given: ΔABC, where $A(-2, 3)$, $B(6, 5)$, and $C(8, 1)$
P is a point on the x-axis for which the sum of the squares of the distances to the vertices of ΔABC has a <u>minimum</u> value. Symbolically, $P(x, 0)$ is the point for which $\left(PA\right)^2 + \left(PB\right)^2 + \left(PC\right)^2$ has a <u>minimum</u> value of N.
Compute the ordered pair (x, N).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014 SOLUTION KEY

Round 3

A) $\left(x^2-16x+\quad\right)+\left(y^2+10y+\quad\right)=11 \Rightarrow$

$\left(x^2-16x+8^2\right)+\left(y^2+10y+5^2\right)=11+64+25=100$

$\Rightarrow \left(x-8\right)^2+\left(y^2+5\right)^2=10^2 \Rightarrow$ Center: **<u>(8, –5)</u>** Radius: **<u>10</u>**

B) Given $A(-3,1)$ and a slope of $-\frac{3}{2}$, in point-slope form, $L_1:(y-1)=-\frac{3}{2}(x+3) \Leftrightarrow 3x+2y=-7$.

Given $B(4,3)$ and $C(0,7)$, the midpoint of $\overline{BC}$ is $(2,5)$ and the slope of $\overline{BC}$ is $\frac{7-3}{0-4}=-1$.

Thus, L_2, the perpendicular bisector of $\overline{BC}$, has a slope of +1, and

$L_2:(y-5)=+1(x-2) \Leftrightarrow x-y=-3$.

Solving $\begin{matrix} L_1: 3x+2y=-7 \\ L_2: x-y=-3 \end{matrix}$ simultaneously, (x, y) = $\underline{\left(-\frac{\mathbf{13}}{\mathbf{5}},\frac{\mathbf{2}}{\mathbf{5}}\right)}$.

C) Given: $A(-2, 3)$, $B(6, 5)$, and $C(8,1)$ For $P(x, 0)$,

$PA^2+PB^2+PC^2=(x+2)^2+9+(x-6)^2+25+(x-8)^2+1=3x^2-24x+139$

Factoring out a 3 and completing the square,

$3\left(x^2-8x\ \underline{+16}\right)+139\ \underline{-48}=3(x-4)^2+91$

Since, for $x \neq 4$, $3(x-4)^2>0$, $x = 4$ produces the minimum value of $91 \Rightarrow (x,N)$ = **<u>(4, 91)</u>**.

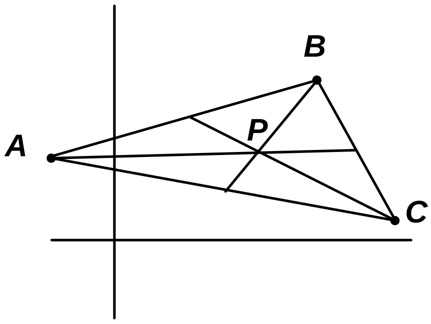
This problem generalizes nicely for an arbitrary point $P(x, y)$.

Show that $PA^2+PB^2+PC^2=3\left((x-4)^2+(y-3)^2\right)+64$, so the minimum value is 64 and it occurs for the point $P(4, 3)$.

This point is the centroid of ΔABC, the point of intersection of the three medians of ΔABC.

This result generalizes to <u>any</u> triangle!

In general, how is the minimum value determined?



Specifically, for ΔABC with vertices $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$, the minimum value of

$$PA^2+PB^2+PC^2=3\left(\left(x-\frac{x_1+x_2+x_3}{3}\right)^2+\left(y-\frac{y_1+y_2+y_3}{3}\right)^2\right)+k$$

Describe how to compute k in terms of x_1, x_2 and x_3.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014
ROUND 4 ALG 2: LOG & EXPONENTIAL FUNCTIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Solve for $x > 0$. $\log_2 \sqrt{x^2 + x^2 + x^2 + x^2} = 5$

B) Compute <u>all</u> possible real values of x for which $125^x + 4(25)^x - 13(5)^x - 52 = 0$.
Express your answer(s) in terms of $\log_5$.

C) Solve for x over the reals: $\dfrac{x}{4} = 2^{\log_x 8}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014 SOLUTION KEY

Round 4

A) $\log_2 \sqrt{x^2+x^2+x^2+x^2} = \log_2 \sqrt{4x^2} = 5 \Rightarrow \sqrt{4x^2} = 2^5 \Rightarrow 4x^2 = 2^{10} \Rightarrow x = \sqrt{\frac{2^{10}}{4}} = \frac{2^5}{2} = \underline{\mathbf{16}}$.

B) Factoring $125^x + 4(25)^x - 13(5)^x - 52 = 0$, we have
$(25)^x\left[5^x+4\right] - 13\left[5^x+4\right] = \left[25^x - 13\right]\left[5^x+4\right] = 0 \Rightarrow 25^x = 13$ ($5^x + 4 > 0$ for all x).
Thus, $5^{2x} = 13 \Rightarrow x = \underline{\mathbf{\frac{\log_5 13}{2}}}$ or $\underline{\mathbf{\log_5 \sqrt{13}}}$.

C) Given: $\frac{x}{4} = 2^{\log_x 8}$
Taking $\log_2$ of both sides, $\log_2 x - \log_2 4 = \log_x 8 = 3\log_x 2$.
$\Leftrightarrow \log_2 x - 2 = \frac{3}{\log_2 x}$
$\Leftrightarrow (\log_2 x)^2 - 2(\log_2 x) - 3 = 0$
$\Leftrightarrow (\log_2 x - 3)(\log_2 x + 1) = 0$
$\Leftrightarrow x = 2^3,\ 2^{-1}$
Thus, $x = \underline{\mathbf{8, \frac{1}{2}}}$.

Alternate Solution (Norm Swanson – Hamilton Wenham)
Let $p = \log_2 x$. Then $x = 2^p$.
Converting the original equation, $\frac{x}{4} = 2^{\log_x 8} = 2^{3\log_x 2} = 2^{\frac{3}{\log_2 x}} \Leftrightarrow \frac{2^p}{4} = 2^{p-2} = 2^{3/p}$.
Equating exponents, $p - 2 = \frac{3}{p} \Rightarrow p^2 - 2p - 3 = 0 \Leftrightarrow (p-3)(p+1) = 0$.
Therefore, $p = 3, -1$ and the solution follows.

What is <u>WRONG</u> with the following "solution"?

$$x = 4\left(2^{\log_x 8}\right) = 4\left(2^{3\log_x 2}\right) = 4\left(2^{\frac{1}{\log_2 x}}\right)^3 = 4\left(2^{(\log_2 x)^{-1}}\right)^3 = 4\left(2^{\log_2 x}\right)^{-3} = 4x^{-3} \Rightarrow x^4 = 4 \Rightarrow x = +\sqrt{2}$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014
ROUND 5 ALG 1: RATIO, PROPORTION OR VARIATION

ANSWERS

A) ________________________________

B) (______________ , ______________)

C) ________________________________

A) For some positive integer values of n, the value of $\frac{174-37n}{4n-3}$ is not an integer.
Compute the minimum value of n for which this is the case.

B) F varies jointly as a and the sum of b and c, and inversely as the square of d.
The proportionality constant is k.
If $F=96$ when $(a,b,c,d)=(80, 5, 7, 4)$ and
$F=50$ when $d=12$ and $a:b:c=1:2:3$, compute the ordered pair (k,c).

C) I have 56 hits in 172 at-bats for a rounded average of 0.326 (hits per at-bats). If I have at least 400 at-bats this season, what is the minimum number of additional hits I must get to exceed an average of 0.400? A Hall of Fame season indeed!

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014 SOLUTION KEY

Round 5

A) Given: $\frac{174-37n}{4n-3}$

$n = 1 \Rightarrow \frac{137}{1} = 137$

Note:
As n increases by 1, the numerator will decrease by 37 and the denominator will increase by 4.

$n = 2 \Rightarrow \frac{100}{5} = 20$ $n = 3 \Rightarrow \frac{63}{9} = 7$ $n = 4 \Rightarrow \frac{26}{13} = 2$ $n = \underline{\mathbf{5}} \Rightarrow \frac{-11}{17}$

B) We are given that $F = k \cdot \frac{a(b+c)}{d^2}$, where k is the proportionality constant.

Substituting, $96 = k \cdot \frac{80 \cdot 12}{16} = 60k \Rightarrow k = \frac{96}{60} = \frac{8}{5}$. Let $(a,b,c) = (x,2x,3x)$. Then:

$50 = \frac{8}{5} \cdot \frac{x(2x+3x)}{12^2} = \frac{8}{5} \cdot \frac{5x^2}{12^2} = \frac{x^2}{18} \Rightarrow x^2 = 18 \cdot 50 = 9 \cdot 100 \Rightarrow x = 30 \Rightarrow c = 90$

Thus, $(k,c) = \underline{\left(\mathbf{\frac{8}{5}, 90}\right)}$.

C) Suppose I gets h hits in x additional at-bats. To exceed a 0.400 average

$\frac{56+h}{172+x} > \frac{2}{5} \Rightarrow 280 + 5h > 344 + 2x \Rightarrow h > \frac{64+2x}{5}$ and $x + 172 \geq 400 \Rightarrow x \geq 228$.

$x = 228 \Rightarrow h > \frac{64+456}{5} = \frac{520}{5} = 104 \Rightarrow h_{\min} = 105$

As x increases, so does $\frac{64+2x}{5}$ which forces h to increase as well.

Thus, **<u>105</u>** is the minimum.

Check: $h = 104 \Rightarrow \frac{56+104}{400} = \frac{16\cancel{0}}{40\cancel{0}} = \frac{2}{5} = 0.400$

$h = 105 \Rightarrow \frac{56+105}{400} = \frac{161}{400} = 0.4025$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014
ROUND 6 PLANE GEOMETRY: POLYGONS (no areas)

ANSWERS

A) _______________________________

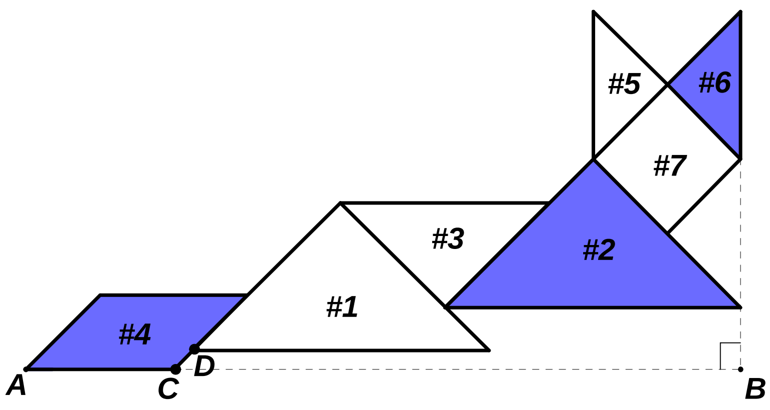
B) _______________________________

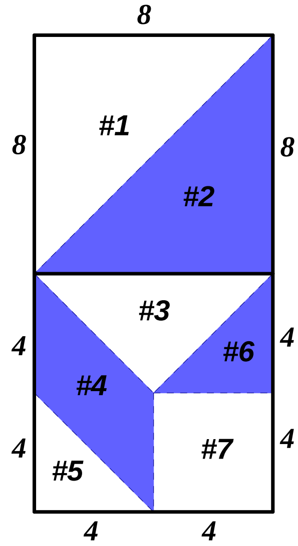
C) (_________ , _________ , _________)

A) O is the center of a regular heptagon (7-gon). P and Q are two vertices for which diagonal $\overline{PQ}$ has a maximum length. If $a^\circ < m\angle POQ < (a+1)^\circ$, compute a.

B) A regular hexagon is partitioned into triangles by the diagonals from a single vertex. The length of a side is $\frac{4}{5}$.
Compute the positive <u>difference</u> between the perimeters of the triangle with the largest perimeter and the triangle with the smallest perimeter.

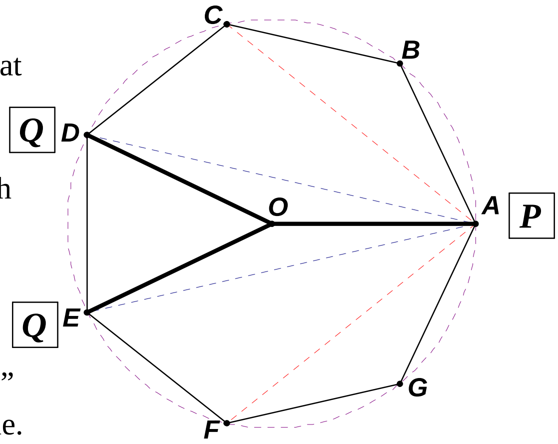
C) A tangram is a puzzle made up of 7 pieces – a square, a parallelogram and 5 isosceles right triangles, formed by dissecting two larger congruent squares, as indicated in the diagram at the right. These pieces can be assembled to form a myriad number of shapes; for example, the cat below. If $CD = 1$, then, as a simplified fraction, $AB = \dfrac{a\sqrt{2}+b}{c}$, where a, b and c are integers. Compute the ordered triple (a,b,c).



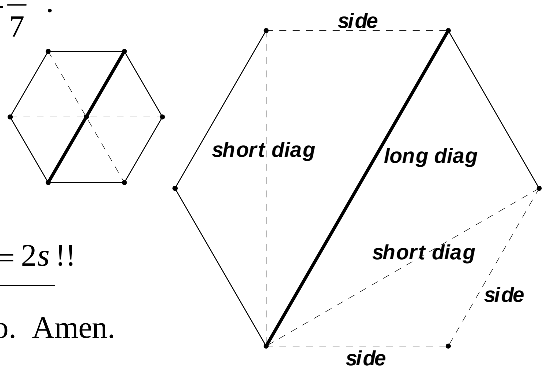


MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014 SOLUTION KEY

Round 6



A) A quick sketch of heptagon *ABCDEFG* will confirm that diagonals $\overline{AC}$ and $\overline{AF}$ have the same length, as do the longer diagonals $\overline{AD}$ and $\overline{AE}$.
There are 2 pairs of congruent diagonals starting at each vertex and ending at some other vertex.
That's total of $\frac{4\cdot 7}{2}=14$ diagonals, 7 short and 7 long.
A short diagonal "skips" one vertex;
a long one "skips" two. As a long diagonal $\overline{PQ}$ "skips" two vertices. Think of the heptagon inscribed in a circle.
Each side determines a central angle at *O* of $\frac{360}{7}^{\circ}$.
$\angle POQ$ consists of 3 of these central angles or $\frac{1080}{7}^{\circ}=154\frac{2}{7}^{\circ}$.
Thus, $a=\underline{\mathbf{154}}$.



B) From the diagram at the right, a triangle is made from either two sides of the hexagon and a short diagonal, or a side of the hexagon, a short diagonal and a long diagonal.
The difference is $\left|(s+d_s+d_l)-(2s+d_s)\right|=\left|d_l-s\right|$, but $\underline{d_l=2s}$!!
The difference is simply $s=\underline{\mathbf{\frac{4}{5}}}$, so we have nothing to do. Amen.

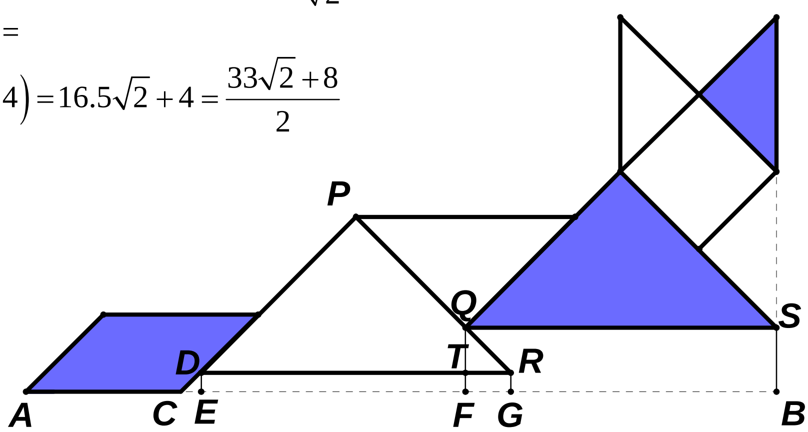
C) For all the pieces, the measures of the interior angles are either 45°, 90° or 135°.
Since the sides of the isosceles right triangle are always in a $1:1:\sqrt{2}$ ratio, we have

$$AC=4\sqrt{2}\ ,CD=1\Rightarrow CE=\frac{\sqrt{2}}{2}\quad EG=DR=8\sqrt{2}$$

$$PR=8,\ PQ=4\sqrt{2}\Rightarrow QR=8-4\sqrt{2}\Rightarrow FG=TR=\frac{QR}{\sqrt{2}}=4\sqrt{2}-4\quad GB=QS-FG=4\sqrt{2}+4$$

$$AB=AC+CE+EG+GB=$$
$$4\sqrt{2}+\frac{\sqrt{2}}{2}+8\sqrt{2}+\left(4\sqrt{2}+4\right)=16.5\sqrt{2}+4=\frac{33\sqrt{2}+8}{2}$$

$$\Rightarrow (a,b,c)=\underline{(\mathbf{33,8,2})}.$$



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________________ D) ____________________________

B) ____________________________ E) _____________ : _____________

C) ____________________________ F) ____________________________

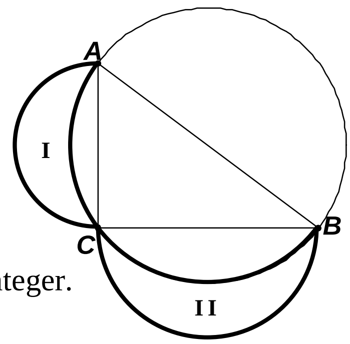
A) In ΔABC, $AB = 8$, $BC = 6$, and D is on $\overline{AB}$ so that $CD = \sqrt{10}$, and $AC = AD$. Compute AC.

B) Two positive reduced fractions $\frac{a}{b}$ and $\frac{c}{d}$, where a, b, c and d are integers and $b \neq d$, have a sum of $\frac{5}{19}$. Compute the <u>minimum</u> sum $b + d$.

C) Circle O is tangent to both $7x + y = 8$ and $x - y = 4$. The radius of circle O is $\sqrt{2}$. The center of circle O is (h, k). Compute <u>all</u> possible values of $h + k$.

D) Suppose P and Q are positive integers and that the point $R\left(4^{2P-Q}, \log_2(P+2Q)\right)$ denotes an ordered pair of positive integers. Compute the <u>smallest</u> possible sum $P + Q$ for which R lies on the line $y = x$.

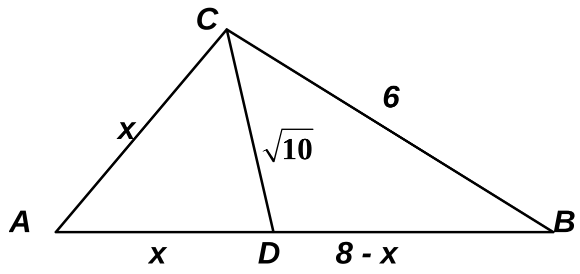
E) A lune is a region bounded by arcs of circles with different radii. The circumcircle of a right triangle with sides of 3, 4 and 5 and semi-circles drawn on the legs in the exterior of the triangle form a pair of lunes. Compute the <u>ratio</u> of the sum of the areas of the lunes (I + II) to the area of the triangle.



F) Regular polygon P has m sides and interior angles of $x°$, where x is an integer. Regular polygon Q has n sides and interior angles of $(x + 2)°$. Compute <u>all</u> possible values of $n - m$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014 SOLUTION KEY

Team Round



A) In ΔABC, $\cos A = \dfrac{8^2 + x^2 - 6^2}{2\cdot 8\cdot x} = \dfrac{28 + x^2}{16x}$.

In ΔACD, $\cos A = \dfrac{x^2 + x^2 - 10}{2\cdot x\cdot x} = \dfrac{x^2 - 5}{x^2}$.

Equating (since $x \neq 0$),
$\dfrac{28+x^2}{16x} = \dfrac{x^2-5}{x^2} \Leftrightarrow \dfrac{28+x^2}{16} = \dfrac{x^2-5}{x}$
Cross multiplying, $28x + x^3 = 16x^2 - 80$ or $x^3 - 16x^2 + 28x + 80 = 0$.

By synthetic division, $\begin{array}{r|rrrr} 4 & 1 & -16 & 28 & 80 \\ & & 4 & -48 & -80 \\ \hline & 1 & -12 & -20 & 0 \end{array}$, we determine that $x = 4$ is a root and, if there are additional roots, then $x^2 - 12x - 20 = 0$. Applying the quadratic formula,
$x = \dfrac{12 \pm \sqrt{144+80}}{2} = \dfrac{12 \pm \sqrt{224}}{2} = \dfrac{12 \pm 4\sqrt{14}}{2} = 6 \pm 2\sqrt{14}$.
$x = 6 + 2\sqrt{14} > 8 \Rightarrow BD < 0$ and must be rejected.
$x = 6 - 2\sqrt{14} < 0$ and must be rejected.
Therefore, the only answer is $AC = \underline{\mathbf{4}}$.
Note that $x = 4$ implies that D was, in fact, a midpoint of $\overline{AB}$, and $\overline{CD}$ was a median.

You could also have verified this result using Stewart's Theorem.

B) Since 19 is prime it is not possible to find two fractions with denominators smaller than 19 that add to $\dfrac{5}{19}$, as it would if the sum were $\dfrac{8}{15}\left(\dfrac{1}{3} + \dfrac{1}{5} = \dfrac{8}{15}\right)$. If the first denominator is k, we have: $\dfrac{5}{19} = \dfrac{a}{k} + \dfrac{m}{n}$ or $\dfrac{5}{19} - \dfrac{a}{k} = \dfrac{m}{n}$. So, $n = 19k$, and the sum of the denominators is $20k$.
Now, to minimize the first denominator, we guesstimate.
$\dfrac{5}{19}$ is smaller than $\dfrac{1}{3}$, but it is just slightly larger than $\dfrac{1}{4}$, so we try $\dfrac{a}{k} = \dfrac{1}{4}$. Then:
$\dfrac{5}{19} - \dfrac{1}{4} = \dfrac{1}{76}$ which gives us $\dfrac{5}{19} = \dfrac{1}{4} + \dfrac{1}{76}$, so the minimum sum of the denominators is $\underline{\mathbf{80}}$.

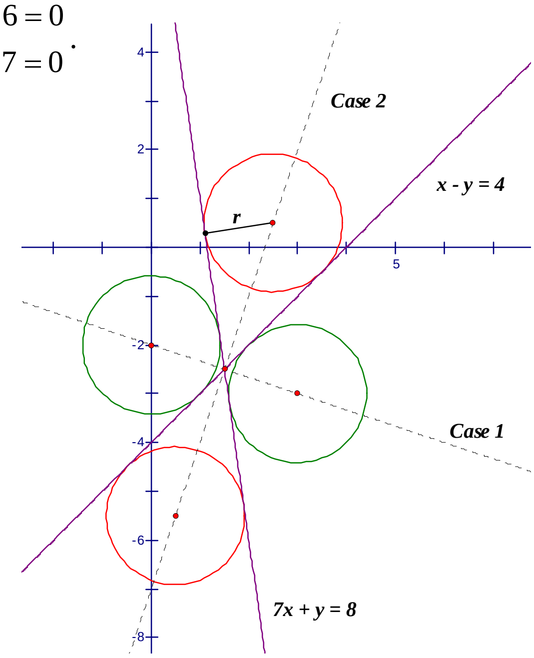
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014 SOLUTION KEY

Team Round – continued

C) The center of circle O lies on one of the angle bisectors of a pair of vertical angles formed by the given lines. The distance from the point (h,k) to the line $Ax+By+C=0$ is given by $\frac{|Ah+Bk+C|}{\sqrt{A^2+B^2}}$. Using the point-to-line distance formula, we have

$$\frac{|7x+y-8|}{\sqrt{7^2+1^2}}=\frac{|x-y-4|}{\sqrt{1^2+(-1)^2}} \Leftrightarrow \frac{|7x+y-8|}{5\cancel{\sqrt{2}}}=\frac{|x-y-4|}{\cancel{\sqrt{2}}} \Leftrightarrow |7x+y-8|=5|x-y-4|$$

$$7x+y-8=\pm 5(x-y-4) \Leftrightarrow \begin{cases} 2x+6y+12=0 \\ 12x-4y-28=0 \end{cases} \Leftrightarrow \begin{cases} x+3y+6=0 \\ 3x-y-7=0 \end{cases}.$$



Case 1: Center on $x + 3y + 6 = 0$ or $y=\frac{-x-6}{3}$

Assume the coordinates of the center is $(h,k)=\left(h,\frac{-h-6}{3}\right)$

Using $7x + y - 8 = 0$ and $\left(h,\frac{-h-6}{3}\right)$, we have

$$\frac{\left|7h+\frac{-h-6}{3}-8\right|}{5\sqrt{2}}=\sqrt{2} \Leftrightarrow \left|\frac{20h}{3}-10\right|=10$$

$$\Leftrightarrow h=\frac{3(10\pm 10)}{20} \Rightarrow h=3,\ 0$$

Thus, $(h,k)=(3,-3),\ (0,-2) \Rightarrow h+k=$ **<u>0, –2</u>**.

Case 2: Center on $3x - y - 7 = 0$ or $y = 3x - 7$

Assume the coordinates of the center is $(h,k)=(h,3h-7)$.

Using $7x + y - 8 = 0$ and $(h,3h-7)$, we have

$$\frac{|7h+(3h-7)-8|}{5\sqrt{2}}=\sqrt{2} \Leftrightarrow |10h-15|=10 \Leftrightarrow |2h-3|=2 \Leftrightarrow 2h=3\pm 2 \Rightarrow h=\frac{5}{2},\ \frac{1}{2}.$$

Thus, $(h,k)=\left(\frac{5}{2},\frac{1}{2}\right),\ \left(\frac{1}{2},-\frac{11}{2}\right) \Rightarrow h+k=$ **<u>3, –5</u>**.

Team Round – continued

D) If R lies on $y = x$, then $4^{2P-Q} = 2^{4P-2Q} = \log_2(P+2Q)$. Taking $\log_2$ of both sides, $4P-2Q = \log_2\left(\log_2\left(P+2Q\right)\right)$. Each expression represents the same positive integer.
$P + 2Q$ must be a power of 2 to insure that $\log_2\left(P+2Q\right)$ is an integer.
$\log_2\left(P+2Q\right)$ must be a power of 2 to insure that $\log_2\left(\log_2\left(P+2Q\right)\right)$ is an integer.
Thus, $P + 2Q$ could be: $2^4 = 16$, $2^8 = 256$, $2^{16} = 65536$, …

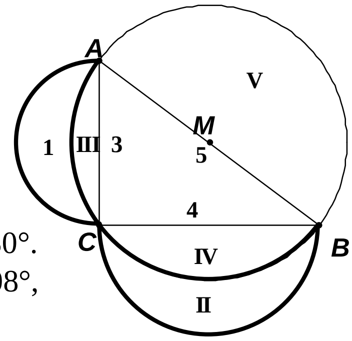
$\begin{cases} P+2Q = 16 \quad 256 \quad 65536 \\ 4P-2Q = \underline{\ 2\ } \quad \underline{\ \ 3\ \ } \quad \underline{\ \ \ \ 4\ } \end{cases} \cdots$ Adding, $5P = \cancel{18}, \cancel{259}, 65540 \Rightarrow (P, Q) = (13108, 26214) \Rightarrow$ **<u>39322</u>**.

E) Note that the Pythagorean Theorem is true for semi-circles (as well as squares), i.e. the area of the semi-circle on $\overline{AB}$ equals the sum of the areas of the semi-circles on $\overline{AC}$ and on $\overline{BC}$.

$\frac{1}{2}\pi\left(\frac{5}{2}\right)^2 = \frac{1}{2}\pi\left(\frac{3}{2}\right)^2 + \frac{1}{2}\pi\left(\frac{4}{2}\right)^2 \Leftrightarrow \frac{\pi}{8}\left(5^2 = 3^2 + 4^2\right)$

Thus, $\left(I + \cancel{III}\right) + \left(II + \cancel{IV}\right) = V = \cancel{III} + \cancel{IV} + \Delta$

$\Rightarrow I + II = \Delta$ and the required ratio is **<u>1 : 1</u>**.



F) As the number of sides of a regular polygon increases, the measure of the interior angles increases with a maximum measure less than 180°. The rate of increase is decelerating. (3,4,5,6, … sides ⇒ 60°, 90°, 108°, 120°, …, differences of 30, 18, 12, … .)

Can we avoid an algebraic blizzard, solving $\begin{cases} \dfrac{180(m-2)}{m} = x° \\ \dfrac{180(n-2)}{n} = (x+2)° \end{cases}$ for ordered pairs (m,n)? Yes!

For a regular polygon with k sides and interior angles of j°, we have $\dfrac{180(k-2)}{k} = j$ or $j = 180 - \dfrac{360}{k}$.
k must be a factor of 360. Examining the factors of 360 in <u>decreasing</u> order produces the largest possible values of j. $360 = 2^3 \cdot 3^2 \cdot 5^1 \Rightarrow (3+1)(2+1)(1+1) = 24 \Rightarrow$ 12 <u>pairs</u> of factors.
Think of pairing the largest factor with the smallest factor, the next largest with the next smallest, etc., namely, $(360,1), (180,2), ..., (20,18) \Rightarrow$

$(k, j) = (360,179), (180,178), (120,177), (90,176), (72,175), (60,174), (45,172),$
$(40,171), (36,170), (30,168), (24,165), (20,162), (18,160), (15,156), (12,150), ...$

Search for j-values which differ by 2 and save the corresponding k-values. By inspection, the possible ordered pairs (m,n) are

$(120,360), (90,180), (72,120), (60,90), (45,60), (36,45), (30,36), (18,20)$ which produces differences of **<u>240, 90, 48, 30, 15, 9, 6</u> and <u>2</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014 ANSWERS

Round 1 Trig: Right Triangles, Laws of Sine and Cosine

A) 450, 468 B) $\left(5,\frac{3\sqrt{2}}{5}\right)$ C) $-\frac{1}{26}$

Round 2 Arithmetic/Elementary Number Theory

A) 6 B) 55 C) 324

Round 3 Coordinate Geometry of Lines and Circles

A) Center: (8, –5) radius: 10 B) $\left(-\frac{13}{5},\frac{2}{5}\right)$ C) (4, 91)

Round 4 Alg 2: Log and Exponential Functions

A) 16 B) $\frac{\log_5 13}{2}$ or $\log_5 \sqrt{13}$ C) 8, $\frac{1}{2}$

Round 5 Alg 1: Ratio, Proportion or Variation

A) 5 B) $\left(\frac{8}{5},90\right)$ C) 105

Round 6 Plane Geometry: Polygons (no areas)

A) 154 B) $\frac{4}{5}$ C) (33, 8, 2)

Team Round

A) 4

B) 80

C) 0, –2, 3, –5

D) 39322

E) 1 : 1

F) 240, 90, 48, 30, 15, 9, 6, 2
(8 values)