MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2015
ROUND 1 ANALYTIC GEOMETRY: ANYTHING

ANSWERS

A) (_____ , _____ , _____ , _____)

B) $(x - \boxed{})^2 + (y - \boxed{})^2 = \boxed{}$

C) (_______ , _______ , _______)

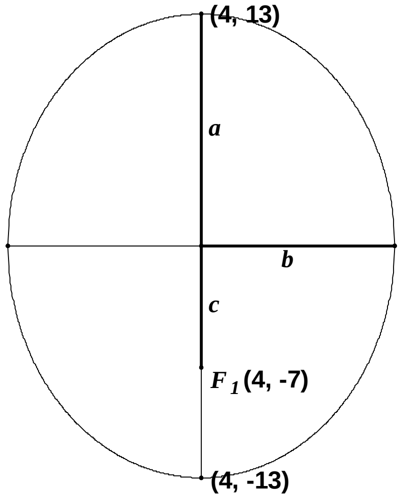
A) An ellipse has a focus at (4, −7) and vertices at (4, ±13).
Its equation is written in the form $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$.
Determine the ordered quadruple (h, k, a^2, b^2).

B) Find the equation of the circle containing the points $P(-5, 2)$, $Q(-3, 4)$ and $R(1, 2)$.
Give your answer in $(x-h)^2 + (y-k)^2 = r^2$ form.

C) The equations of the asymptotes of the conic defined by $9x^2 - 6y^2 + 18x + 18 = 0$ are written in the form $y = \pm m(x-h) + k$. Compute the ordered triple (m, h, k), where $m > 0$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2015 SOLUTION KEY

Round 1



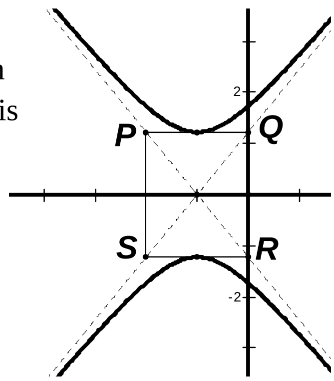
A) Given: ellipse w/focus $F(4, -7)$ and vertices $(4, \pm 13)$.
The ellipse must be vertical w/center at (4, 0), $a = 13$, $c = 7$
For an ellipse, $a^2 = b^2 + c^2 \Rightarrow b^2 = 120$.
For the ellipse, the a-value is <u>always</u> larger than the b-value.
Since the ellipse is vertical, the y-term will have the larger denominator, producing an equation $\frac{(x-4)^2}{120} + \frac{y^2}{169} = 1$.
Thus, $(h, k, a^2, b^2) = \underline{(\mathbf{4, 0, 120, 169})}$.

B) Given: a circle passing through $P(-5, 2)$, $Q(-3, 4)$ and $R(1, 2)$
The equation of any circle can be expressed in the form $x^2 + y^2 + Cx + Dy + E = 0$.
Substituting, we have
1) $-5C + 2D + E = -29$
2) $-3C + 4D + E = -25$
3) $C + 2D + E = -5$

Solving this system of simultaneous equations, $C = 4$, $D = -2$, and $E = -5$
Thus, equation is $x^2 + y^2 + 4x - 2y - 5 = 0$ or, completing the square, $\underline{\mathbf{(x + 2)^2 + (y - 1)^2 = 10}}$.

An alternate solution takes advantage of the fact that chord $\overline{PR}$ is horizontal.
The perpendicular bisector of any chord in a circle passes through the center of the circle.
Talk about this approach with your teammates and/or coach. Some additional hints are at the end of the solution key.

C) $9x^2 - 6y^2 + 18x + 18 = 0 \Leftrightarrow 2y^2 - 3x^2 - 6x - 6 = 0 \Leftrightarrow 2y^2 - 3(x+1)^2 = 3$ or $\boxed{\frac{y^2}{\frac{3}{2}} - \frac{(x+1)^2}{1} = 1}$



Since $y = \pm m(x-h) + k \Leftrightarrow (y-k) = \pm m(x-h)$, we note that the equations of the asymptotes $\overleftrightarrow{PR}$ and $\overleftrightarrow{SQ}$ are in point-slope form. From the equation of the hyperbola (boxed above), we see that the hyperbola is vertical, point (h, k) is the center of the hyperbola and the slopes of the asymptotes are $\pm\frac{a}{b}$. The center is at $(-1, 0)$ and

$a = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}, b = 1 \Rightarrow m = \frac{\sqrt{6}}{2}$. Thus, $(m, h, k) = \underline{\left(\mathbf{\frac{\sqrt{6}}{2}, -1, 0}\right)}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2015
ROUND 2 ALG 1: FACTORING AND/OR EQUATIONS INVOLVING FACTORING

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) The trinomial $24x^2 + 175x + 100$ factors as $(Ax+B)(Cx+D)$.
Compute the product $ABCD$.

B) Compute <u>all</u> real roots of $(x-3)^3 + 2(x-3)^2 = 8x - 24$.

C) Compute all values of x for which $\dfrac{x}{2x+3} - \dfrac{x+1}{3-2x} - \dfrac{20x}{8x^2-18} = 0$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2015 SOLUTION KEY

Round 2

A) Without calculating *A*, *B*, *C* or *D* or even factoring the trinomial!
Since $(Ax+B)(Cx+D)=ACx^2+(AD+BC)x+BD$, equating coefficients, we have AC = 24 and BD = 100. Thus, $ABCD$ = **2400**.
Note: $24x^2+175x+100$ factors as $(8x+5)(3x+20)$ and
$8\cdot 5\cdot 3\cdot 20=(8\cdot 3)(5\cdot 20)=24(100)=2400$

B) $(x-3)^3+2(x-3)^2=8x-24$
$\Leftrightarrow (x-3)^3+2(x-3)^2=8(x-3)$
$\Leftrightarrow (x-3)^3+2(x-3)^2-8(x-3)=0$
Let $a=(x-3)$ and factor out the common binomial term.
$a\left(a^2+2a-8\right)=a(a+4)(a-2)=0\Rightarrow a=0,-4,2\Rightarrow x=$ **3, −1, 5** (in any order).

C) To avoid division by zero, note that $x\neq\pm\frac{3}{2}$

$$\frac{x}{2x+3}-\frac{x+1}{3-2x}-\frac{20x}{8x^2-18}=0\Leftrightarrow \frac{x}{2x+3}+\frac{x+1}{2x-3}-\frac{10x}{(2x+3)(2x-3)}=0$$

Therefore, the least common denominator is $(2x+3)(2x-3)$.
Multiplying through by the LCD, we have $x(2x-3)+(x+1)(2x+3)-10x=0$
$\Leftrightarrow 2x^2-3x+2x^2+5x+3-10x=0$
$\Leftrightarrow 4x^2-8x+3=(2x-1)(2x-3)=0\Rightarrow x=$ $\mathbf{\frac{1}{2}}$, $\cancel{\frac{3}{2}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2015
ROUND 3 TRIG: EQUATIONS WITH A REASONABLE NUMBER OF SOLUTIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

All solutions must be expressed in radians.
Pay attention to the specified range of the solutions in each question.

A) Compute the four solutions for which $x \in [0, 2\pi)$ and $\frac{1}{2} + \sin^2 x = \cos^2 x$.

B) Solve for x over $\underline{-\pi < x < 0}$: $8\cos^3 x - 4\cos^2 x - 2\cos x + 1 = 0$

C) Compute the two values of x over $\underline{0 < x < \frac{\pi}{2}}$: $\tan\left(2x - \frac{\pi}{4}\right) = \cot\left(3x + \frac{\pi}{6}\right)$

MASSACHUSETTS MATHEMATICS LEAGUE
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Round 3

A) $\frac{1}{2}+\sin^2 x=\cos^2 x \Leftrightarrow \cos^2 x-\sin^2 x=\frac{1}{2} \Leftrightarrow \cos 2x=\frac{1}{2}$

Therefore, $2x=\pm\frac{\pi}{3}+2k\pi \Leftrightarrow x=\pm\frac{\pi}{6}+k\pi$

$k=0 \Rightarrow x=\pm\frac{\pi}{6} \Rightarrow \underline{\boldsymbol{\frac{\pi}{6}}}, -\frac{\pi}{6}+2\pi=\underline{\boldsymbol{\frac{11\pi}{6}}}$

$k=1 \Rightarrow x=\pm\frac{\pi}{6}+\pi=\underline{\boldsymbol{\frac{5\pi}{6}}}, \underline{\boldsymbol{\frac{7\pi}{6}}}$

Alternate Solution!! (Mike Schockett – Maimonides)
Adding $\sin^2 x$ to both sides,

$\frac{1}{2}+2\sin^2 x=\sin^2 x+\cos^2 x=1 \Rightarrow \sin^2 x=\frac{1-\frac{1}{2}}{2}=\frac{1}{4} \Rightarrow \sin x=\pm\frac{1}{2} \Rightarrow \frac{\pi}{6}$ - family.

B) A solution interval of $-\pi<x<0$ implies that the solutions must be in quadrants 3 and 4.

$$8\cos^3 x-4\cos^2 x-2\cos x+1=0$$

Grouping the first two terms and the last two terms, $4\cos^2 x(2\cos x-1)-(2\cos x-1)=0$.

$\Leftrightarrow (4\cos^2 x-1)(2\cos x-1)=0$

$\Leftrightarrow (2\cos x-1)^2(2\cos x+1)=0 \Rightarrow \cos x=\pm\frac{1}{2} \Rightarrow x=\underline{\boldsymbol{-\frac{\pi}{3}, -\frac{2\pi}{3}}}$

Synthetic division could also have been used with the coefficients of $8, -4, -2, 1$, producing a double root of $+\frac{1}{2}$ and a single root of $-\frac{1}{2}$ and the solution follows as above.

C) A solution interval of $\underline{0<x<\frac{\pi}{2}}$ implies that all solutions must be in quadrant 1.

$\tan\left(2x-\frac{\pi}{4}\right)=\cot\left(3x+\frac{\pi}{6}\right) \Rightarrow \tan\left(2x-\frac{\pi}{4}\right)=\tan\left(\frac{\pi}{2}-\left(3x+\frac{\pi}{6}\right)\right)$

Since $\tan A=\tan B \Rightarrow A=B\pm k\pi$, we have $2x-\frac{\pi}{4}=\frac{\pi}{2}-\left(3x+\frac{\pi}{6}\right)\pm k\pi$

$\Rightarrow 5x=\frac{\pi}{2}-\frac{\pi}{6}+\frac{\pi}{4}\pm k\pi=\frac{7\pi}{12}\pm k\pi \Rightarrow x=\frac{(7\pm 12k)\pi}{60}$ and $k=0, 1 \Rightarrow x=\underline{\boldsymbol{\frac{7\pi}{60}, \frac{19\pi}{60}}}$

Even if we had not been given that there were only 2 solutions, $k=2 \Rightarrow x=\frac{31\pi}{60}>\frac{\pi}{2}$ and we stop searching.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2015
ROUND 4 ALG 2: QUADRATIC EQUATIONS

ANSWERS

A) (_____________ , _____________)

B) ______________________________

C) ______________________________

A) Given: $x = y^2 - 10y + 3$
The <u>minimum</u> value of *x*, call it *m*, occurs for *y* = *k*.
Compute the ordered pair (*m*, *k*).

B) For what values of the constant *m* does the following <u>quadratic</u> equation have roots that are real and unequal?

$$2m^2x^2 - 7mx = -3m(1 + x)$$

C) Find <u>all</u> values of the constant *k* for which the equation $(k+7)x^2 - kx + 9 = 0$ has exactly one solution.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2015 SOLUTION KEY

Round 4

A) Completing the square on the right side,
$x = y^2 - 10y + 3 \Leftrightarrow x = \left(y^2 - 10y + 25\right) + 3 - 25 \Leftrightarrow (y-5)^2 - 22$
For all values of *y*, $\left(y-5\right)^2 \geq 0$. Therefore, the minimum value of *x* (namely *m*) is –22 and it occurs when *y* = 5. The required ordered pair is $\left(m,k\right) = \underline{\left(\mathbf{-22,5}\right)}$.

B) $2m^2x^2 - 7mx = -3m\left(1+x\right) \Leftrightarrow 2m^2x^2 - 4mx + 3m = 0 \Leftrightarrow m\left(2mx^2 - 4x + 3\right) = 0$
Examining the discriminant of the quadratic factor, $16 - 24m > 0 \Leftrightarrow m < \frac{2}{3}$ (but the quadratic equation disappears for *m* = 0). Thus, $\underline{\boldsymbol{m < \frac{2}{3}, m \neq 0}}$.

C) If *k* = –7, the equation becomes linear and has only one solution.
If $k \neq -7$, the equation is quadratic and, if the discriminant equals zero, there will be exactly one solution. $k^2 - 36(k+7) = 0 \Rightarrow k^2 - 36k - 252 = (k+6)(k-42) = 0$.
Thus, there are three values, *k* = **<u>42, –6, –7</u>** (in any order).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2015
ROUND 5 GEOMETRY: SIMILARITY OF POLYGONS

ANSWERS

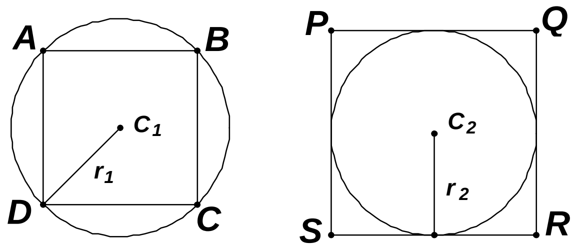
A) _____________ : _______________

B) _______________________________

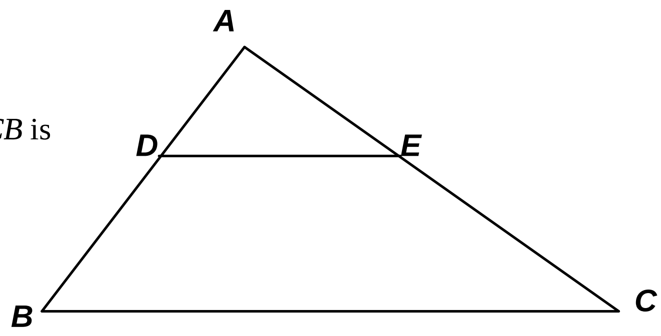
C) _____________ : _______________

A) *ABCD* and *PQRS* are squares.
ABCD is inscribed in circle C_1 with radius $r_1 = 2$.
Circle C_2 with radius $r_2 = 2$ is inscribed in *PQRS*.

Compute the ratio of the perimeter of *ABCD* to the perimeter of *PQRS*.

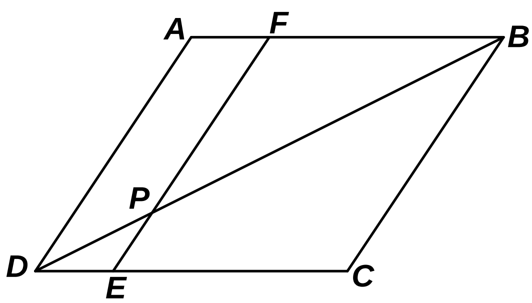


B) Given: $\overline{DE} \parallel \overline{BC}$, $AD = 4$, $AE = 6$ and $DE = 8$
If the ratio of the area of ΔADE to the area of *DECB* is 4 : 21, compute *BC*.



C) *ABCD* is a rhombus.
$\overline{EF} \parallel \overline{AD}$, $\overline{BD} \cap \overline{EF} = \{P\}$, and $\dfrac{area(\Delta DPE)}{area(ADPF)} = \dfrac{1}{6}$

Compute $\dfrac{area(\Delta BPF)}{area(CEPB)}$.



MASSACHUSETTS MATHEMATICS LEAGUE
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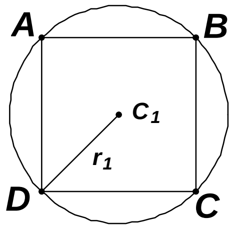
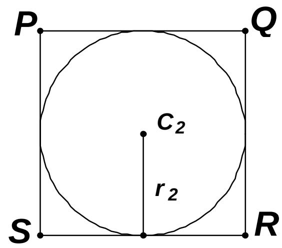
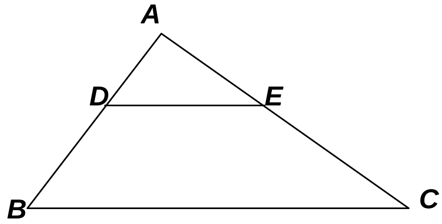
Round 5



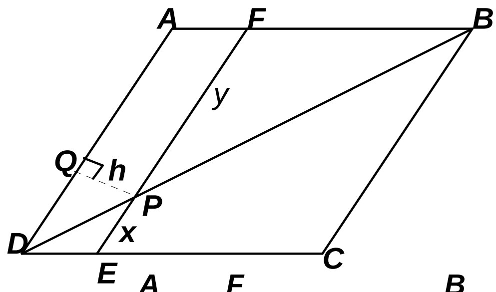


A) The diagonal of *ABCD* = $2r_1 = 4$ and its side must be $\frac{4}{\sqrt{2}} = 2\sqrt{2}$. In *PQRS*, $PS = 2r_2 = 4$. Thus, the perimeters are in the same ratio as the sides, namely, $\frac{2\sqrt{2}}{4} = \underline{\boldsymbol{\frac{\sqrt{2}}{2}}}$.



B) If the area of ΔADE : area of *DECB* is 4 : 21, then the area of ΔADE : area of ΔABC is 4 : (4 + 21) = 4 : 25 which implies the ratio of corresponding sides is 2 : 5.
Thus, $\frac{8}{BC} = \frac{2}{5} \Rightarrow BC = \underline{\mathbf{20}}$.

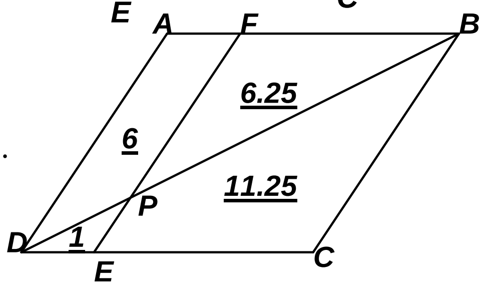
C) Let $h = PQ$ denote the distance between the parallels $\overline{AD}$ and $\overline{EF}$.



$$\frac{area(\Delta DPE)}{area(ADPF)} = \frac{\cancel{\frac{1}{2}}hx}{\cancel{\frac{1}{2}}h\left(y+(x+y)\right)} = \frac{1}{6}$$

$$\Rightarrow \frac{x}{x+2y} = \frac{1}{6} \Rightarrow 6x = x + 2y \Rightarrow \frac{x}{y} = \frac{2}{5}$$

$$\Delta DPE \sim \Delta BPF \Rightarrow \frac{area(\Delta DPE)}{area(\Delta BPF)} = \left(\frac{2}{5}\right)^2 = \frac{4}{25}$$



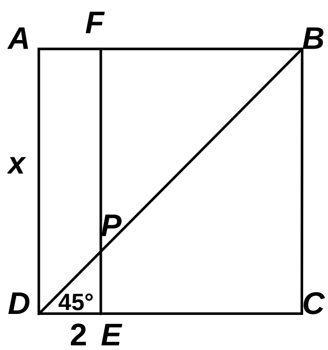
If the area of ΔDPE be 1, then $area(\Delta BPF) = \frac{25}{4} = 6.25$.

Since $\Delta BAD \cong \Delta DCB$, $area(CEPB) = 11.25$ and the required ratio is $\frac{6.25}{11.25} = \frac{25}{45} = \underline{\boldsymbol{\frac{5}{9}}}$.

Alternate Solution (Norm Swanson – Hamilton-Wenham - retired)
Assume *ABCD* is a square. (A square is a rhombus.) Let *DE* = *EP* = 2, and *AD* = *x* ⇒ *PF* = *x* – 2 ⇒ area(ΔDPE) = 2 ⇒ area(*ADPF*) = 12

$$\frac{1}{2}(2)\left(x+(x-2)\right) = 12 \Rightarrow x = AD = 7 \Rightarrow PF = FB = 5$$

Thus, the required ratio is $\frac{12.5}{35-12.5} = \frac{25}{70-25} = \underline{\boldsymbol{\frac{5}{9}}}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2015
ROUND 6 ALG 1: ANYTHING

ANSWERS

A) (_____________ , _____________)

B) ________________________________

C) ________________________________

A) Compute the ordered pair (A, B) for which $2x^2 + A(x-B) = (2x-3)(x+6)$ is an identity, that is, a statement which is true for all values of x.

B) In Primordia, the coins are worth 2¢, 5¢, 17¢ and 31¢ (according to the most recent exchange rate). I asked a clerk at the 7-11 Store to give me change for $1.00, but I could not resist adding that I wanted a minimum number of coins. He gave me 9 coins – three 2¢ coins, three 5¢ coins, one 17¢ coin and two 31¢ coins. He should have been able to do better. What is the <u>minimum</u> number of coins he should have given me?

C) Today is Linda's and Sam's birthday. Linda said to Sam, "Comparing our ages 10 years ago, I was 1 year less than twice your age.". Sam said to Linda, "Your age in 13 years will be $\frac{5}{6}$ of my age in 29 years.". In how many years from today will the ratio of Linda's age to Sam's age be 6 : 5 ?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2015 SOLUTION KEY

Round 6

A) $2x^2 + A(x-B) = (2x-3)(x+6) \Leftrightarrow 2x^2 + Ax - AB = 2x^2 + 9x - 18$
If these two trinomials are to give the same values for every value of *x*, then
$A = 9$ and $AB = 18 \Rightarrow (A,B) = \underline{\mathbf{(9,2)}}$.

B) Using as many of the larger denominations as possible first will minimize the number of coins needed.
$\left\lfloor \frac{100}{31} \right\rfloor = 3 \Rightarrow 100 - 31 \cdot 3 = 7$ which can be returned as one 2¢ coin and one 5¢ coin.
Therefore, only **<u>5</u>** coins are needed.

C) Let Linda and Sam be *x* and *y* years old today. Then:
$x - 10 = 2(y-10) - 1 \Rightarrow x = 2y - 11$ and
$x + 13 = \frac{5}{6}(y+29) \Rightarrow 6x - 5y = 67$
Substituting for *x*, $6(2y-11) - 5y = 67 \Rightarrow 7y = 133 \Rightarrow (x,y) = (27,19)$
Suppose the required 6 : 5 ratio occurs in *T* years. Then:
$\frac{27+T}{19+T} = \frac{6}{5} \Leftarrow 135 + 5T = 114 + 6T \Rightarrow T = \underline{\mathbf{21}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2015
ROUND 7 TEAM QUESTIONS

ANSWERS

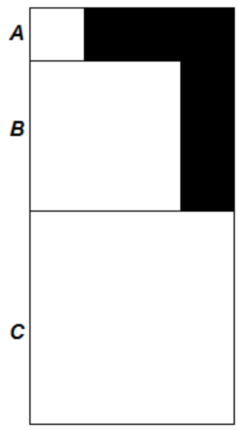
A) ______________________ D) ______________________

B) ______________________ E) ______________________

C) ____________________________________ F) __________

____

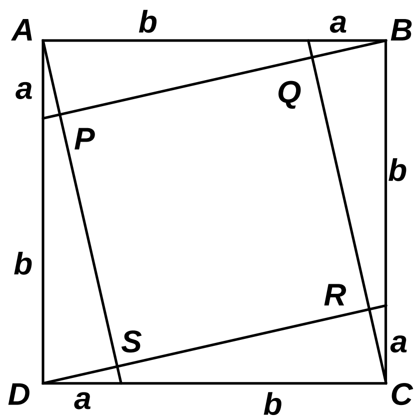
A) Consider the parabola P whose equation is $y = 2x^2 - 8x + C$.
The graph of P passes through the point $A(1, 1)$.
The line $\overleftrightarrow{AF}$, where F is the focus of P, intersects P in a second point Q.
Compute the x-coordinate of Q.



B) Compute <u>all</u> possible ordered pairs of real numbers (x, y) for which
$$y = \sqrt{x-1} \text{ and } x = \sqrt{72y+1}.$$

C) Specify all values (in radians) over $0 < x < 2\pi$ for which $\left|8\sin^2 x - 5\right| < 1$.

D) For positive integers A, B and C, $A^2 + B^2 = C^2$, where $B > A$ and $C = B + 2$. Three squares with <u>sides</u> A, B and C are stacked as in the diagram at the right.
The area of the shaded region is 910.
Compute $A + B + C$.



E) The ratio of the area of square $ABCD$ to the area of the square $PQRS$ can be expressed as $1+\dfrac{2k}{b^2}$.
Determine an expression for k in terms of a and b.

F) 16, $5a - 4$ and a^2 are three <u>distinct</u> rational numbers.
For specific values of a, one of these rational numbers is the arithmetic mean of the other two. Compute the arithmetic mean of <u>all</u> the possible a-values.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2015 SOLUTION KEY

Team Round

A) (1, 1) on P ($y = 2x^2 - 8x + C$) $\Rightarrow 1 = 2(1)^2 - 8(1) + C \Rightarrow C = 7$

$y = 2x^2 - 8x + 7 \Leftrightarrow y - 7 = 2(x-2)^2 - 8 \Leftrightarrow (x-2)^2 = \frac{1}{2}(y+1)$ (Parabola opens UP.)

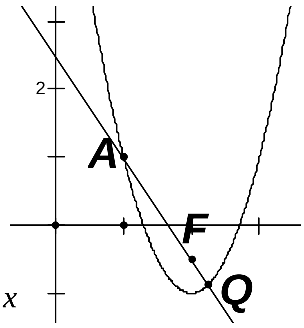
Thus, the vertex of the parabola is at (2, –1) and $4p = \frac{1}{2} \Rightarrow p = \frac{1}{8} \Rightarrow$ focus $F\left(2, -\frac{7}{8}\right)$.

The slope of $\overleftrightarrow{AF}$ is $\dfrac{1+\frac{7}{8}}{1-2} = -\frac{15}{8}$ and the equation of $\overleftrightarrow{AF}$ is

$(y-1) = -\frac{15}{8}(x-1)$ or $y = \frac{-15x+23}{8}$. Knowing that $x = 1$ would be a root of the following quadratic equation helps factor the trinomial.

Substituting, $\frac{-15x+23}{8} = 2x^2 - 8x + 7 \Leftrightarrow 16x^2 - 49x + 33 = 0 \Leftrightarrow (16x-33)(x-1) = 0$

$\Rightarrow x_Q = \underline{\mathbf{\frac{33}{16}}}$.



B) Note the original equations $y = \sqrt{x-1}$ and $x = \sqrt{72y+1}$ require that both x and y be nonnegative. Since, in the first equation, $x = 1 \Rightarrow y = 0$ and (1, 0) satisfies the second equation, we have the (trivial) solution **<u>(1, 0)</u>**.

Squaring both sides, we have $\begin{cases} y^2 = x - 1 \Leftrightarrow x = y^2 + 1 \\ x^2 = 72y + 1 \end{cases}$.

Substituting for x in the second equation, $(y^2+1)^2 = 72y + 1 \Leftrightarrow y^4 + 2y^2 - 72y = 0$

If $y \neq 0$, $y^3 + 2y - 72 = 0$.

By inspection (lucky guess) or synthetic substitution, y = 4 is solution.

Synthetic substitution gives the complete factorization as $(y-4)(y^2+4y+18)$ and the trinomial factor does not give additional real solutions.

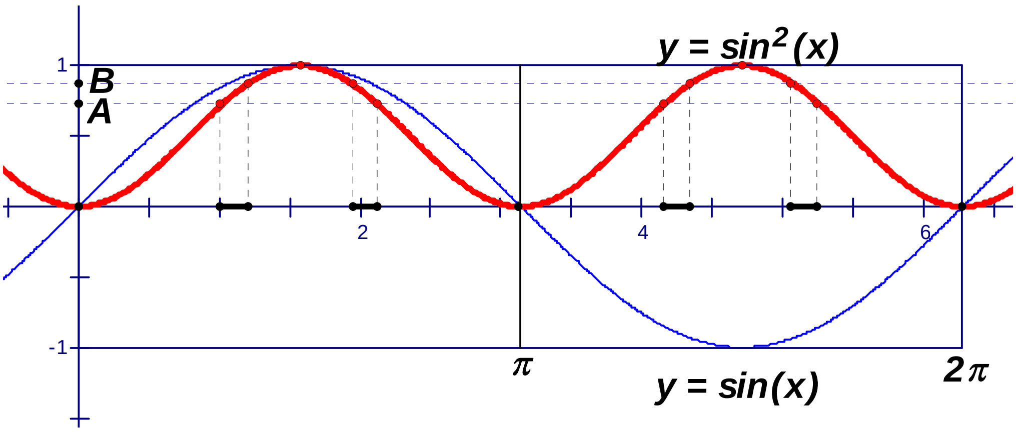
$y = 4 \Rightarrow 16 = x - 1 \Rightarrow x = 17$ and a second solution is the ordered pair $\underline{(\mathbf{17,4})}$.

Team Round - continued

C) $\left|8\sin^2 x-5\right|<1 \Leftrightarrow -1<8\sin^2 x-5<1 \Leftrightarrow \frac{1}{2}<\sin^2 x<\frac{3}{4} \Rightarrow \frac{\sqrt{2}}{2}<\sin x<\frac{\sqrt{3}}{2}$

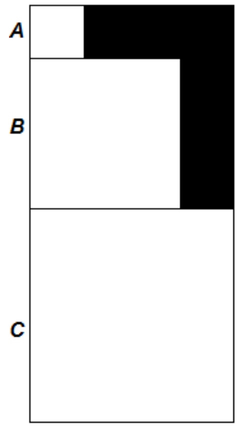
As a visual clue, the graphs of $y=\sin(x)$ and $y=\sin^2(x)$ are included below.
The graph of the latter is either on or above the *x*-axis.
The graphs share the same *x*-intercepts and the same maximum value, but notice that between 0 and π, the graph of $y=\sin^2(x)$ is below $y=\sin(x)$, since squaring a value between 0 and 1 produces a smaller value.

Let $A\left(0,\frac{\sqrt{2}}{2}\right)$ and $B\left(0,\frac{\sqrt{3}}{2}\right)$ The horizontal lines through *A* and *B* intersect the graph of $y=\sin^2(x)$ in 8 points over the specified interval $0\le x<2\pi$. Since $\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{3}}{2}$ are special values, we recognize the *x*-values (i.e. the coordinates of the endpoints of the solution intervals) as the $\frac{\pi}{4}$ and $\frac{\pi}{3}$ families of related values.



The solution is 4 disjoint intervals, namely
$\underline{\boldsymbol{\frac{\pi}{4}<x<\frac{\pi}{3}}}$, $\underline{\boldsymbol{\frac{2\pi}{3}<x<\frac{3\pi}{4}}}$, $\underline{\boldsymbol{\frac{5\pi}{4}<x<\frac{4\pi}{3}}}$ and $\underline{\boldsymbol{\frac{5\pi}{3}<x<\frac{7\pi}{4}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2015 SOLUTION KEY



Team Round - continued

D) The area of the entire region is

$A^2 + B^2 + C^2 + 910 = C(A+B+C) = AC + BC + C^2$

Cancelling and transposing terms, $910 = (AC + BC) - (A^2 + B^2)$

Factoring and substituting for $A^2 + B^2$, $C(A+B) - C^2 = 910$

$\Rightarrow C((A+B)-C) = C(A-(C-B)) = (B+2)(A-2) = 910$

Examining a table of Pythagorean Triples, where the difference between the lengths of the hypotenuse and the longer leg is 2:
A is increasing by 2, the *B*-gap increases by 2 each time, *C* = *B* + 2
Every other row is a ***primitive*** Pythagorean Triple.

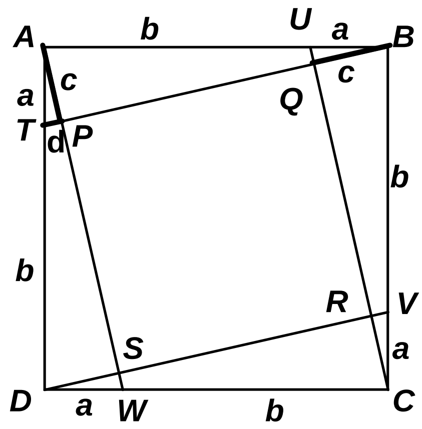
	A	B	C	(B+2)(A-2)	****
1	6	8	10	10·4	40
2	8	15	17	17·6	102
3	10	24	26	26·8	208
4	12	35	37	37·10	370
5	14	48	50	50·12	600
6	16	63	65	65·14	**910**
7	18	80	82	17·6	
8	20	99	101		

Thus, $A+B+C$ = 16 + 63 + 65 = **144**. Note (as a check) that the area of the rectangle as the sum of the areas of 4 regions $16^2 + 63^2 + 65^2 + 910 = 256 + 3969 + 4225 + 910 = 9360$ and as a length times width computation $65(16+63+65) = 65(144) = 9360$ are equal.

Solution #2 (Norm Swanson – Hamilton Wenham - retired)
Alternately, without resorting to a table, we could think of $(B+2)(A-2) = 910 = 2\cdot5\cdot7\cdot13$ as $\frac{(B+2)(A-2)}{10} = 7\cdot13$. Since $B > A$, we try $\frac{B+2}{5} = 13$ and $\frac{A-2}{2} = 7$.
This gives us $B = 65 - 2 = 63$ and $A = 14 + 2 = 16$ and $(A,B,C) = (16,63,65)$ works!
Think about the second step.
A keen number sense inspired this insight, along with the recognition that 7 and 13 are primes.
[In honesty, we could have considered $91 = 13\cdot7$ or $91\cdot1$ and $10 = 5\cdot2$ or $10\cdot1$.]
None of the other possibilities produces a *C*-value which satisfies the P.T. Check it out.

The formula for the areas listed in the rightmost column of the chart above is $2(n+1)(n^2+4n+5)$ which can be written as $2(n+1)\left[(n+2)+i\right]\left[(n+2)-i\right]$. $n = 6 \Rightarrow 2\cdot7\cdot(8+i)(8-i) = 14\cdot65 = 910$
If the *B/C* relation were $C = B+1$ (instead of $C = B+2$), the first few triples and areas would be $(3,4,5,10)$, $(5,12,13,52)$, $(7,24,25,150)$,… . Can you determine a formula for the area?

Team Round - continued



E) $\Delta TAP \sim \Delta WAD \Rightarrow \quad \frac{short}{hypot} = \frac{TP}{DW} = \frac{d}{a} = \frac{a}{\sqrt{(a+b)^2 + a^2}}$

$$\Rightarrow d = \frac{a^2}{\sqrt{(a+b)^2 + a^2}}$$

Also, $\frac{long}{hypot} = \frac{AP}{AW} = \frac{c}{a} = \frac{a+b}{\sqrt{(a+b)^2 + a^2}}$

$$\Rightarrow c = \frac{a(a+b)}{\sqrt{(a+b)^2 + a^2}}$$

Therefore, the side of the square $PS = AW - (c + d)$

$$= \sqrt{(a+b)^2 + a^2} - \left(\frac{a^2}{\sqrt{(a+b)^2 + a^2}} + \frac{a(a+b)}{\sqrt{(a+b)^2 + a^2}} \right)$$

Expressing with a common denominator, this is $\frac{\left((a+b)^2 + a^2\right) - a^2 - a(a+b)}{\sqrt{(a+b)^2 + a^2}}$

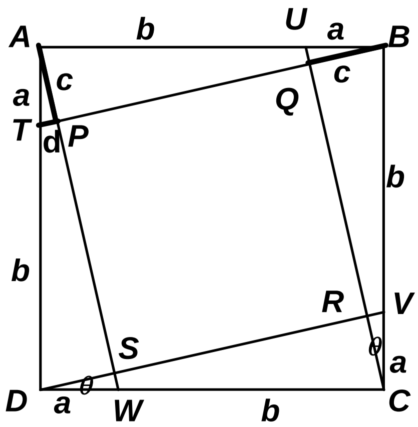
Expanding and simplifying the numerator, this is $\frac{b(a+b)}{\sqrt{(a+b)^2 + a^2}}$

Finally, the required ratio is

$$\frac{area(ABCD)}{area(PQRS)} = \frac{(a+b)^2}{\left(\frac{b(a+b)}{\sqrt{(a+b)^2 + a^2}} \right)^2} = \frac{(a+b)^2 \left((a+b)^2 + a^2\right)}{b^2 (a+b)^2} = \frac{a^2 + (a+b)^2}{b^2} = 1 + \frac{2\left(a^2 + ab\right)}{b^2}$$

$\Rightarrow k = \underline{\boldsymbol{a^2 + ab}}$ or equivalent.

Team Round - continued



E) Alternate Approach using Trigonometry
(Norm Swanson – Hamilton Wenham – retired)

Since the ratio of the areas of square *ABCD* to square *PQRS* is $\left(\frac{DC}{SR}\right)^2$, let's find *DC* and *SR* in terms of $m\angle SDW = m\angle RCV = \theta$.

In ΔDCV , $\cos\theta = \frac{DC}{DV} = \frac{a+b}{DV}$ and $\sin\theta = \frac{VC}{DV} = \frac{a}{DV}$.

$\Rightarrow\ DV = (a+b)\sec\theta = a\csc\theta$.

Expanding, $b\sec\theta = a(\csc\theta - \sec\theta) \Rightarrow b = a\left(\frac{\csc\theta}{\sec\theta} - 1\right) \Rightarrow \boxed{b = a(\cot\theta - 1)}$.

In ΔDSW , $DS = a\cos\theta$. In ΔRVC , $RV = a\sin\theta$.

Therefore, $\frac{DC}{SR} = \frac{a+b}{DV - DS - RV} = \frac{a + \boxed{a(\cot\theta - 1)}}{a(\csc\theta - \cos\theta - \sin\theta)} = \frac{\cot\theta}{\csc\theta - \cos\theta - \sin\theta}$

$$= \frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta} - \cos\theta - \sin\theta} = \frac{\cos\theta}{1 - \sin\theta\cos\theta - \sin^2\theta} = \frac{\cos\theta}{\left(\underbrace{1 - \sin^2\theta}_{\cos^2\theta}\right) - \sin\theta\cos\theta} = \frac{\cancel{\cos\theta}}{\cancel{\cos\theta}(\cos\theta - \sin\theta)}$$

Squaring this ratio, we have $\left(\frac{DC}{SR}\right)^2 = \frac{1}{\cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta} = \frac{1}{1 - 2\sin\theta\cos\theta} = \frac{1}{1 - \sin 2\theta}$

Converting to the required form might follow these lines:

$$\frac{1}{1 - \sin 2\theta} = \frac{1}{1 - 2\frac{CV}{DV}\cdot\frac{CD}{DV}} = \frac{DV^2}{DV^2 - 2CV \cdot CD}$$

Substituting for DV^2, using the Pythagorean Theorem on ΔDCV , we have

$$\frac{DC^2 + CV^2}{DC^2 + CV^2 - 2CV \cdot DV} = \frac{DC^2 + CV^2}{(DC - CV)^2} = \frac{(a+b)^2 + a^2}{b^2} = 1 + \frac{2(a^2 + ab)}{b^2} \Rightarrow k = \underline{\boldsymbol{a^2 + ab}}\ .$$

Team Round - continued

F) Case 1: 16 is the mean

$$16=\frac{5a-4+a^2}{2}\Rightarrow a^2+5a-36=0\Rightarrow(a+9)\cancel{(a-4)}=0\ \Rightarrow\ \boxed{a=-9}$$

(If a = 4, then all three numbers are the same.)

Case 2: $5a-4$ is the mean

$$5a-4=\frac{16+a^2}{2}\Rightarrow a^2-10a+24=0\Rightarrow(a-6)\cancel{(a-4)}=0\ \Rightarrow\ \boxed{a=6}$$

Case 3: a^2 is the mean

$$a^2=\frac{16+5a-4}{2}\Rightarrow 2a^2-5a-12=0\Rightarrow(2a+3)\cancel{(a-4)}=0\ \Rightarrow\ \boxed{a=-\frac{3}{2}}$$

The mean is $\dfrac{-9+6+\left(-\frac{3}{2}\right)}{3}\cdot\dfrac{2}{2}=\dfrac{-6-3}{6}=\underline{\boldsymbol{-\frac{3}{2}}}$.

Round 1 Question C

Since the midpoint of $\overline{PR}$ is $(-2,2)$ and the perpendicular is vertical, we have $h=-2$ and the center O is at $(-2,k)$. Since OR = OQ = r, we can use the distance formula to find k, and then substitute to find r. The details are left to you.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2015 ANSWERS

Round 1 Analytic Geometry: Anything

A) $(4,0,120,169)$ B) $(x+2)^2+(y-1)^2=10$ C) $\left(\frac{\sqrt{6}}{2},-1,0\right)$

Round 2 Alg: Factoring

A) 2400 B) –1, 3, 5 C) $\frac{1}{2}$

Round 3 Trig: Equations

A) $\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$ B) $-\frac{\pi}{3},-\frac{2\pi}{3}$ C) $\frac{7\pi}{60},\frac{19\pi}{60}$

Round 4 Alg 2: Quadratic Equations

A) (–22, 5) B) $m<\frac{2}{3}, m\neq 0$ C) 42, –6, –7
All 3 answers are required.

Round 5 Geometry: Similarity

A) $\frac{\sqrt{2}}{2}$ B) 20 C) 5 : 9

Round 6 Alg 1: Anything

A) (9, 2) B) 5 C) 21

Team Round

A) $\frac{33}{16}$

B) (1, 0), (17, 4)

C) $\frac{\pi}{4}<x<\frac{\pi}{3}$, $\frac{2\pi}{3}<x<\frac{3\pi}{4}$,
$\frac{5\pi}{4}<x<\frac{4\pi}{3}$, $\frac{5\pi}{3}<x<\frac{7\pi}{4}$
(All 4 intervals required.)

D) 144

E) a^2+ab (or equivalent)

F) $-\frac{3}{2}$