MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2015
ROUND 1 ALG 2: SIMULTANEOUS EQUATIONS AND DETERMINANTS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Compute the <u>largest</u> integer value of *k* for which the determinant $\begin{vmatrix} 3 & -5k \\ 4 & k+100 \end{vmatrix}$ is negative.

B) For what value(s) of *c* will the following system of equations have an infinite number of solutions?

$$\begin{aligned} 14x+3y-7z&=8 \\ -8x+5y+4z&=c \\ -2x+3y+z&=2 \end{aligned}$$

C) Given: $A(10,-7)$, $B(-6,11)$, and $C(3,k)$

The area of ΔABC is given by the formula $\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$, that is, <u>half</u> the absolute value of the determinant of the 3 x 3 matrix formed by the coordinates of the vertices, taken clockwise (or counterclockwise) with 1s filling the third column.

Determine <u>all</u> <u>single-digit</u> positive integral values of *k* for which the area of ΔABC is an integer perfect square.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2015 SOLUTION KEY

Round 1 (Additional comments on 1B at the end of the solution key)

A) $\begin{vmatrix} 3 & -5k \\ 4 & k+100 \end{vmatrix} = 3(k+100) - 4(-5k) = 23k + 300 < 0 \Leftrightarrow k < \frac{-300}{23} = -13\frac{1}{23} \Rightarrow$ **$\underline{-14}$**

B) Given:
(1) $14x + 3y - 7z = 8$
(2) $-8x + 5y + 4z = c$ (3) $\Rightarrow z = 2 + 2x - 3y$
(3) $-2x + 3y + z = 2$

Substituting in (1), $14x + 3y - 14 - 14x + 21y = 8 \Rightarrow 24y = 22 \Rightarrow \begin{cases} y = \frac{11}{12} \\ z = 2x - \frac{3}{4} \end{cases}$

Substituting in (2), $-8x + \frac{55}{12} + 8x - 3 = c \Rightarrow c =$ **$\underline{\frac{19}{12}}$**, otherwise, there is no solution.

<u>FYI</u>: Provided $c = \frac{19}{12}$, the solution set is $\left\{\left(x, \frac{11}{12}, 2x - \frac{3}{4}\right)\right\}$. The equations become $\begin{cases} \frac{11}{4} + \frac{21}{4} = 8 \\ \frac{55}{12} - 3 = c \\ \frac{11}{4} - \frac{3}{4} = 2 \end{cases}$

as the *x*-terms drop out. The 3 equations are satisfied if and only if $c = \frac{19}{12}$.

Note that if the *z*-coefficients are multiplied by –2, they equal the *x*-coefficients.

Therefore, the 3 equations are dependent and $c = \frac{19}{12}$ makes the system consistent.

Alternately, since the columns are dependent, the determinant of the matrix of coefficients is zero, and, for there to be an infinite number of solutions, the determinant of the matrix of coefficients, where any column is replaced by the constants, must also be zero.

For example, $\begin{vmatrix} 8 & 3 & -7 \\ c & 5 & 4 \\ 2 & 3 & 1 \end{vmatrix} = (40 + 24 - 21c) - (-70 + 96 + 3c) = 0 \Rightarrow 38 - 24c = 0 \Rightarrow c = \frac{19}{12}$.

Replacing the *z*-coefficients gives the same *c*-value. If the *y*-coefficients are replaced, the determinant of the matrix is zero, for <u>all</u> values of *c*; so again 0/0 ⇒ an infinite number of solutions.

C) $\begin{matrix} 10 & -7 & 1 & 10 & -7 \\ -6 & 11 & 1 & -6 & 11 \\ 3 & k & 1 & 3 & k \end{matrix} \Rightarrow A = \frac{1}{2}\left|\det\begin{bmatrix} 10 & -7 & 1 \\ -6 & 11 & 1 \\ 3 & k & 1 \end{bmatrix}\right|$

$= \frac{1}{2}\left|\left(10\cdot 11\cdot 1 + (-7)\cdot 1\cdot 3 + 1\cdot(-6)\cdot k\right) - \left(3\cdot 11\cdot 1 + k\cdot 1\cdot 10 + 1\cdot(-6)\cdot(-7)\right)\right|$

$= \frac{1}{2}\left|110 - 21 - 6k - 33 - 10k - 42\right| = \frac{1}{2}\left|14 - 16k\right| = \left|7 - 8k\right|$.

$\left|7 - 8k\right|$ evaluates to a perfect square for k = **<u>1</u>**, **<u>2</u>**, **<u>4</u>**, and **<u>7</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2015
ROUND 2 ALG1: EXPONENTS AND RADICALS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) Determine the <u>largest</u> integer value of k for which $\frac{-2^4+(-8)^2}{2^k}$ is an integer.

B) $\left(\sqrt{5}-\sqrt{2}\right)^4 = A - B\sqrt{10}$, where A and B are integers.
Compute $(A-B)^2$.

C) Given: $(x+y)^2 = 225(x-y)^{-2}$, where $\underline{x < y < 0}$.
Compute all possible <u>integer</u> ordered pairs (x, y).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2015 SOLUTION KEY

Round 2

A) $-2^4+(-8)^2=-\left(2^4\right)+(-8)^2=-16+64=48=2^4\cdot 3\Rightarrow k=\underline{\mathbf{4}}$

REMINDER (if you thought -2^4 was 16):
Consider that $-2^4=0-2^4$ and PEMDAS requires exponentiation be done before subtraction.
There are no parentheses!

B) $\left(\sqrt{5}-\sqrt{2}\right)^4=\left(\left(\sqrt{5}-\sqrt{2}\right)^2\right)^2=\left(7-2\sqrt{10}\right)^2=89-28\sqrt{10}=A-B\sqrt{10}\Rightarrow(A,B)=(89,28)$

Thus, $(A-B)^2=61^2=(60+1)^2=3600+120+1=\underline{\mathbf{3721}}$.

C) $(x+y)^2=225(x-y)^{-2}\Rightarrow (x+y)^2(x-y)^2=225\Rightarrow\left(x^2-y^2\right)^2=225\Rightarrow x^2-y^2=\pm 15$

Since $x<y<0$, we need consider only $x^2-y^2=+15$
Also, for $x<y<0$, we have $x+y<x-y$.

$\Rightarrow\begin{cases}x+y=-15\\x-y=-1\end{cases}$ or $\begin{cases}x+y=-5\\x-y=-3\end{cases}$

Solving, the first set of equations gives us $\underline{\mathbf{(-8,-7)}}$; the second $\underline{\mathbf{(-4,-1)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2015
ROUND 3 TRIGONOMETRY: ANYTHING

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) The solution set to a trig equation is denoted $5\theta = \begin{cases} 60° + n \cdot 360° \\ 120° + n \cdot 360° \end{cases}$, where n is an integer.

Compute the <u>sum</u> of the degree-measures of the <u>obtuse</u> values of θ belonging to the solution set.

B) Solve for x, where $0 < x < 2\pi$. $\left|\tan x + \sqrt{3}\right| \left(\cos x + \frac{\sqrt{3}}{2}\right)^5 \left(\sin x + \frac{3}{2}\right)^3 < 0$

C) For $x = 5\pi/8$, compute $1 + \sin^2 x + \cos^2 x + \tan^2 x + \cot^2 x + \sec^2 x + \csc^2 x$.

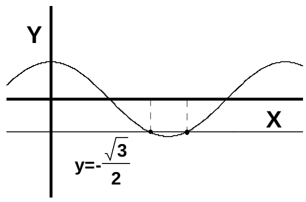
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2015 SOLUTION KEY

Round 3

A) $5\theta = \begin{cases} 60° + n \cdot 360° \\ 120° + n \cdot 360° \end{cases} \Leftrightarrow \theta = \begin{cases} 12° + n \cdot 72° \\ 24° + n \cdot 72° \end{cases} \Rightarrow \begin{cases} 12, 84, 156 \\ 24, 96, 168 \end{cases} \Rightarrow 96 + 156 + 168 = \underline{\mathbf{420}}$

B) The first factor is 0 for $x = \frac{2\pi}{3}$ and $\frac{5\pi}{3}$, so these values are excluded.

Only the middle factor can be negative! $\cos x < -\frac{\sqrt{3}}{2}$



$\frac{\sqrt{3}}{2} \Rightarrow \frac{\pi}{6}$ family, Q2, 3 $\Rightarrow$ open interval

$\underline{\mathbf{\frac{5\pi}{6} < x < \frac{7\pi}{6}} \text{ or } \mathbf{\left(\frac{5\pi}{6}, \frac{7\pi}{6}\right)}.}$

Note: The excluded values for the first factor are outside this interval.

C) Simplify the given expression before attempting to substitute.

$1 + \left(\sin^2 x + \cos^2 x\right) + \tan^2 x + \cot^2 x + \sec^2 x + \csc^2 x = \sec^2 x + \csc^2 x + \sec^2 x + \csc^2 x$

$= 2\left(\sec^2 + \csc^2 x\right)$

$= 2\left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right) = 2\left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\right) = \frac{2}{\frac{1}{4}\left(2\sin x \cos x\right)^2} = \frac{8}{\sin^2 2x}$

Now we can substitute, $\frac{8}{\left(\sin\frac{5\pi}{4}\right)^2} = \frac{8}{\left(-\frac{\sqrt{2}}{2}\right)^2} = \underline{\mathbf{16}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2015
ROUND 4 ALG 1: ANYTHING

ANSWERS

A) ____________________________mph

B) (________ , ________ , ________)

C) ________________________________

A) Yesterday I drove from San Antonio to Houston, Texas.
It took me one hour.
For 20 minutes, I drove at an average speed of 60 mph.
For the next 20 minutes, I increased my average speed by 50%.
For the last 20 minutes, I decreased my average speed to 50% of my average speed for the second 20 minute interval.
What was my average speed for the hour?

B) The maximum value of a quadratic function $y = ax^2 + bx + c$ is 6 when $x = 2$. This function has a y-intercept of 4. Determine the ordered triple (a, b, c)

C) The inequality $k \le |x-2| \le 10$, where k is an integer, has exactly 10 integer solutions. Compute k.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2015 SOLUTION KEY

Round 4

A) For each 20 minute period, my speeds were 60 mph, 90 mph and 45 mph, during which I travelled 20 miles, 30 miles and 15 miles. Therefore, I travelled a total of 65 miles in the hour, for an average speed of **65** mph.

B) Since the maximum value of the function is 6 for $x = 2$, the graph of the function must pass through the point $P(2, 6) \Rightarrow y = ax^2 + bx + c$ must be equivalent to $y = a(x-2)^2 + 6$ and Since the y-intercept is 4, the function passes through the point $Q(0, 4)$

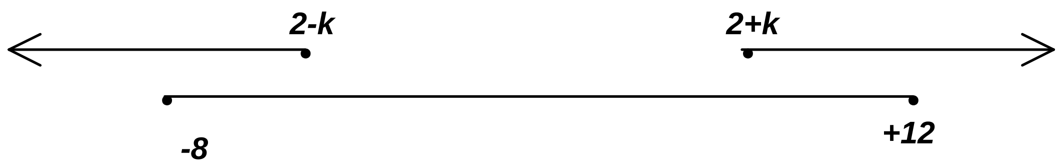
$\Rightarrow 4 = a(0-2)^2 + 6 = 4a + 6 \Rightarrow a = -\frac{1}{2}$

$y = -\frac{1}{2}(x-2)^2 + 6 = -\frac{1}{2}x^2 + 2x + 4 \Rightarrow (a,b,c) = \underline{\left(-\frac{1}{2}, 2, 4\right)}$.

C) $k \le |x-2| \le 10 \Rightarrow |x-2| \le 10$ and $|x-2| \ge k$

$|x-2| \le 10 \Rightarrow -10 \le x-2 \le +10 \Rightarrow -8 \le x \le 12$ (outer limits)

$|x-2| \ge k \Rightarrow x-2 \le -k$ or $x-2 \le +k \Rightarrow x \le 2-k$ or $x \le k+2$ (inner limits)



Therefore, the solution in general consists of two segments of length 10 – k.
There will be the same number of solution on both segments, i.e. 5 each.
The five on the right must be 8, 9, 10, 11 and 12. Thus, k = **6**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2015
ROUND 5 PLANE GEOMETRY: ANYTHING

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) A regular polygon with 135 diagonals is inscribed in a circle.
What is the maximum degree-measure of an angle formed by two diagonals drawn from the same vertex?

B) An obtuse triangle has two sides of lengths 10 and 15. Find the number of possible integral lengths of the third side.

C) Circle O has two perpendicular chords $\overline{AB}$ and $\overline{CD}$ of length 16 and 19 respectively, intersecting at point P in the interior of the circle. PA and PC are integers. Compute the largest possible radius r for the circle O.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2015 SOLUTION KEY

Round 5

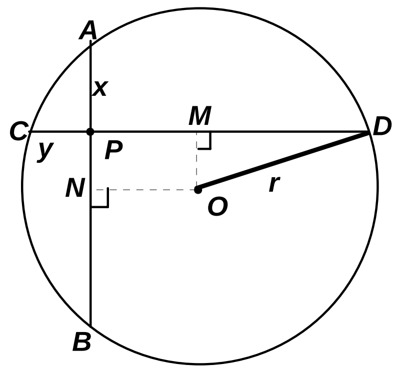
A) Since $d=\dfrac{n(n-3)}{2}=135$, by trial and error, we have $\dfrac{18\cdot 15}{2}=9\cdot 15=135$ and n = 18.
Thus, there are 18 sides and 15 diagonals at each vertex.
The 18 vertices divide the circle into 18 congruent arcs, each measuring 20°.
The 1^{st} and 15^{th} diagonal at any vertex form an inscribed angle consisting of 14 of these 18 arcs. Its measure is $\dfrac{1}{2}(14\cdot 20)=\underline{\mathbf{140}°}$.

B) If x is the longest side, $x < 10 + 15 = 25$.
If it were a right triangle, x would be the hypotenuse and $10^2 + 15^2 = x^2$. In an obtuse triangle, $x^2 > 10^2 + 15^2 = 325$. So, we have a lower limit for x; specifically, since $18^2 = 324$, $x \geq 19$.
In this case $19 \leq x \leq 24$ (6 possibilities)

If 15 is the longest side, $x + 10 > 15 \Rightarrow x \geq 6$ and $x^2 + 100 < 225$.
$x^2 < 125 \Rightarrow x \leq 11$.
In this case, $6 \leq x \leq 11$ (6 possibilities)
Total: **<u>12</u>**

C) Given: $(AB, CD) = (16,19)$

Let x and y denote the lengths of $\overline{PA}$ and $\overline{PC}$ respectively.
According to the product chord theorem, $x(16-x)=y(19-y)$ where x and y are integers. Without this restriction there are infinitely many solutions. Examine the products of the lengths of each chord.



on $\overline{AB}$: (1, 15) - 15, (2, 14) - 28, (3,13) - 39, (4, 12) - **48**, (5, 11) - 55, (6, 10) - **60**, (7, 9) - 63, (8, 8) - 64
on $\overline{CD}$: (1, 18) - 18, (2, 17) - 34, (3, 16) – **48**, (4, 15) – **60**, (5, 14) – 70, (6,13) – 78, (7,12) – 84, (8,11) – 88 (9,10) - 90
$\overline{ON}$ bisects $\overline{AB}$ and $\overline{OM}$ bisects $\overline{CD}$.
$x = 4 \Rightarrow NP = OM = 8 - 4 = 4$
$MD = 9.5$. In ΔMOD, $r^2 = 4^2 + 9.5^2 = 16 + \dfrac{19^2}{4} = \dfrac{64+361}{4} = \dfrac{425}{4} = \dfrac{25\cdot 17}{4} \Rightarrow r = \underline{\mathbf{\dfrac{5}{2}\sqrt{17}}}$.
For $x = 6$, we get $r^2 = 2^2 + 9.5^2$ and clearly this r-value will be smaller.

See additional comments on the <u>original</u> 5C at the end of the solution key.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2015
ROUND 6 ALG 2: PROBABILITY AND THE BINOMIAL THEOREM

ANSWERS

A) ______________________________

B) ______________________________

C) (____________ , ____________)

A) Five fair coins are tossed. Compute the probability that the result will be 3 heads and two tails or one head and 4 tails.

B) The expansion of $\left(4x^2+k\right)^3$ contains the term $432x^2$ and the constant term is negative. Compute the value of *k*.

C) The formula $\dfrac{n!}{k!\cdot(n-k)!}$ works nicely for evaluating coefficients in the binomial expansion of $\left(A+B\right)^n$, when *n* is an integer. If *n* is a fraction, factorials are undefined, but we can use the following equivalent $\dfrac{n(n-1)(n-2)\cdot\ldots\cdot(n-k+1)}{1\cdot2\cdot3\cdot\ldots\cdot k}$, where both the numerator and denominator contain *k* factors. This formula can be used to expand $\sqrt{a+x}=\left(a+x\right)^{\frac{1}{2}}$.
This is particularly useful when *x* is very much less than *a* so that the first few terms provide a good approximation of the value of the expression.
The first three terms in the expansion of $\sqrt{1+x}$ are $1+\dfrac{1}{2}x-\dfrac{1}{8}x^2$.
The next two terms are $\underline{Ax^3-Bx^4}$.
Compute the ordered pair (*A*, *B*), each entry being a ratio of relatively prime integers.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2015 SOLUTION KEY

Round 6

A) The probability of {H,H,H,T,T} is $\frac{5!}{3!2!}\left(\frac{1}{2}\right)^5 = \frac{10}{32}$.

The probability of {H,T,T,T,T} is $\frac{5!}{1!4!}\left(\frac{1}{2}\right)^5 = \frac{5}{32}$.

Since there is no overlap, the probability of either/or is $\underline{\mathbf{\frac{15}{32}}}$.

B) Using the 3rd row of Pascal's triangle (1-3-3-1), in the expansion of $\left(4x^2+k\right)^3$, the x^2-term is

$3\cdot\left(4x^2\right)^1\cdot k^2 = 432x^2 \Rightarrow 12k^2 = 432 \Rightarrow k^2 = 36 \Rightarrow k = \pm 6$.

However, the constant term is negative, so k = **<u>–6</u>** only.
The same results could have been obtained by cubing the binomial by brute force.

C)

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \frac{\frac{1}{2}}{1!}x + \frac{\frac{1}{2}\cdot-\frac{1}{2}}{2!}x^2 + \frac{\frac{1}{2}\cdot-\frac{1}{2}\cdot-\frac{3}{2}}{3!}x^3 + \frac{\frac{1}{2}\cdot-\frac{1}{2}\cdot-\frac{3}{2}\cdot-\frac{5}{2}}{4!}x^4 = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \boxed{\frac{\frac{3}{8}}{6}x^3 + \frac{-\frac{15}{16}}{24}x^4}$$

$$\Rightarrow (A,B) = \underline{\left(\mathbf{\frac{1}{16}},\mathbf{\frac{5}{128}}\right)}$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2015
ROUND 7 TEAM QUESTIONS
ANSWERS

A) ____________________________ D) **(______ , ______ , ______)**

B) ____________________________ E) ____________________________

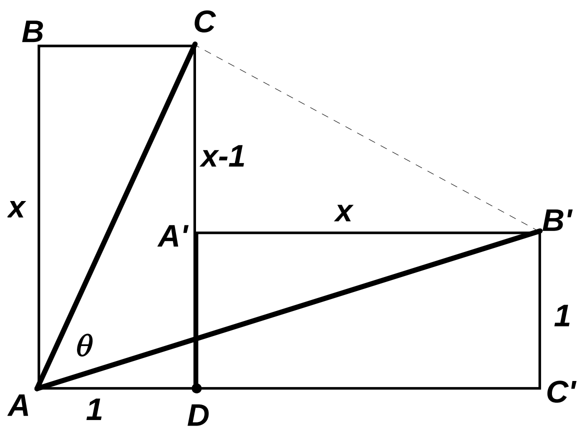
C) ____________________________ F) ____________________________

A) There are four ordered pairs (x, y) that satisfy $\begin{cases} x^2 - xy + y^2 = 7 \\ \dfrac{4}{x} + 3y = 1 \end{cases}$.

Let x_1, x_2, x_3, x_4 denote the x-coordinates of these 4 ordered pairs.
Let a, b, c, d denote 4 integers, where $b < c < d$.
$x_1 = a$, but x_2, x_3, x_4 are irrational numbers satisfying
the inequalities $\begin{cases} b < x_2 < b+1 \\ c < x_3 < c+1 \\ d < x_4 < d+1 \end{cases}$. Compute ad.

B) Compute the two integer values of x for which
$$4^{2x+a} = 8^{5-bx}, \text{ if } a : b = 2 : 1$$

C) *ABCD* is a 1 by x rectangle, where $x > 1$ is an integer.
ABCD is rotated 90° clockwise about point *D* to a new position. Compute $B'C$ for which $m\angle CAB'$ is closest to 60°.



D) For nonzero real constants a and b, the linear equations $y = ax + b$ and $\dfrac{x}{a} - \dfrac{y}{b} = 1$ intersect.
P, the point of intersection, is not on the line $y = x$, if $a = \underline{p}$, or if $a \neq p$ and $b \neq \underline{q}a^2 + \underline{r}a$, for constants p, q and r. Compute the ordered triple (p, q, r).

E) The lengths of two sides of a parallelogram *ABCD* are x and $x + c$, where x and c are both positive integers. The lengths of the diagonals are $x + 3$ and $x + 5$. Find all possible perimeters of parallelogram *ABCD*.

F) Urn #1 contains 1 white, 2 red and 3 blue balls. Urn #2 contains 4 white, 4 red and 2 blue balls.
The balls are indistinguishable except for color.
Two balls are drawn simultaneously from urn #1 and added to urn #2.
After the draw from urn #1, there are x white balls, y red balls and z blue balls remaining in urn #1 and $x \neq y, y \neq z$, and $x \neq z$.
Two balls are now simultaneously drawn from urn #2.
Compute the probability that these balls will be the same color.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2015 SOLUTION KEY

Team Round

A) Solving the second equation for *y*, $y = \frac{x-4}{3x}$.

Substituting in the first equation,

$$x^2 - \frac{x-4}{3} + \frac{x^2-8x+16}{9x^2} = 7 \Rightarrow 9x^4 - 3x^2(x-4) + x^2 - 8x + 16 = 63x^2 \Rightarrow 9x^4 - 3x^3 - 50x^2 - 8x + 16 = 0$$

We know that there is an integer root! By synthetic division $\begin{array}{r|rrrrr} & 9 & -3 & -50 & -8 & 16 \\ \hline -2 & 9 & -21 & -8 & 8 & 0 \end{array}$, we confirm *x* = –2 is a root and the quotient is $9x^3 - 21x^2 - 8x + 8$.

Continuing synthetic division of this quotient, we watch for the remainder to change sign.

$$\begin{array}{r|rrrr} & 9 & -21 & -8 & 8 \\ \hline -1 & 9 & -30 & 22 & -14 \\ 0 & 9 & -21 & -8 & 8 \\ 1 & 9 & -12 & -20 & -12 \\ 2 & 9 & -3 & -14 & -20 \\ 3 & 9 & 6 & 10 & 38 \end{array}$$

Thus, we have roots in the intervals (–1, 0), (0, 1) and (2, 3).

This gives us *b* = –1, *c* = 0, *d* = 2, and the required product *ad* = **<u>–4</u>**.

B) $4^{2x+a} = 8^{5-bx} \Leftrightarrow 2^{4x+2a} = 2^{15-3bx}$

Equating the exponents, transposing terms and solving for *x*, $4x + 2a = 15 - 3bx \Rightarrow x = \frac{15-2a}{3b+4}$

Substituting $a = 2b$, $x = \boxed{\frac{-4b+15}{3b+4}} = \frac{-4+\frac{15}{b}}{3+\frac{4}{b}}$. Clearly, $b = -1$ gives division by –1 and *x* will be an integer, namely $\frac{-19}{-1} = \mathbf{\underline{19}}$. For even values of *b*, the numerator of the boxed expression is odd, while the denominator is even, and *x* will not be an integer.

Looking at the complex fraction expression for *x*, as $b \to \pm\infty$, we see that $x \to -\frac{4}{3}$.

So we must consider "small" odd values of *b*.

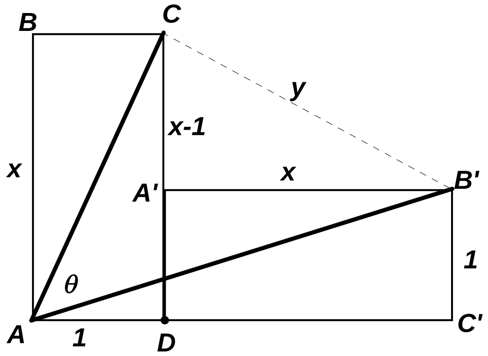
Wanting to avoid mindless plug and chug, we try long division,

$$\begin{array}{r} -4/3 \\ (3b+4)\overline{)\,-4b+15} \\ \underline{-4b-16/3} \\ 61/3 \end{array}$$

, getting $x = -\frac{4}{3} + \frac{61}{3(3b+4)}$. Since 61 is prime, we have limited choices for a divisor.

If $b = 19$, then $x = -\frac{4}{3} + \frac{1}{3} = \mathbf{\underline{-1}}$.

Team Round - continued



C) Using the Pythagorean Theorem,

$y^2 = x^2 + (x-1)^2$

$(AC)^2 = x^2 + 1$

$(AB')^2 = (x+1)^2 + 1$

Using the law of cosines on $\Delta AB'C$,

$y^2 = \left(x^2+1\right) + \left((x+1)^2+1\right) - 2\sqrt{\left(x^2+1\right)\left((x+1)^2+1\right)}\cos 60°$

$2x^2 - 2x + 1 = 2x^2 + 2x + 3 - \sqrt{\left(x^2+1\right)\left((x+1)^2+1\right)}$

Transposing,

$\sqrt{\left(x^2+1\right)\left((x+1)^2+1\right)} = 4x + 2$

$\left(x^2+1\right)\left(x^2+2x+2\right) = \left(4x+2\right)^2$

$x^4 + 2x^3 + 3x^2 + 2x + 2 = 16x^2 + 16x + 4$

$x^4 + 2x^3 - 13x^2 - 14x - 2 = 0$

Using synthetic substitution with integer values of $x > 1$, we look for the closest value to zero.

	1	2	-13	-14	-2
2	1	4	-5	-24	-50
3	1	5	2	-8	-26
4	1	6	11	30	118

Clearly, $x = 3$ produces the angle closest to 60° and $B'C = \underline{\sqrt{\mathbf{13}}}$.

Team Round – continued

D) For $a=1$, $L_1\left(y=ax+b\right)$ and $L_2\left(\frac{x}{a}-\frac{y}{b}=1\right)$ do not intersect on $L_3\left(y=x\right)$, since L_1 is parallel to L_3 for nonzero values of *b*. Therefore, all ordered pairs $\left(a,b\right)$, where $a=1$ must be excluded, regardless of the value of *b*, and we have $p=1$.

How about when $a\neq 1$? Substituting for *y* in the second equation,

$$\left.\begin{array}{l} y=ax+b \\ \frac{x}{a}-\frac{y}{b}=1 \end{array}\right\} \Rightarrow bx-ay=ab$$

$$\Rightarrow bx-a\left(ax+b\right)=ab$$

$$\Rightarrow x\left(b-a^2\right)=2ab \Rightarrow \boxed{x=\frac{2ab}{b-a^2}}$$ Substituting in the first equation for *x*,

$$y=a\left(\frac{2ab}{b-a^2}\right)+b=\frac{2a^2b+b^2-a^2b}{b-a^2}=\frac{a^2b+b^2}{b-a^2}$$

Thus, the point of intersection is $\left(\frac{2ab}{b-a^2},\frac{a^2b+b^2}{b-a^2}\right)$.

Since, we were given that L_1 and L_2 intersect, $b-a^2\neq 0$.

P can only be on L_3, if $2ab=a^2b+b^2$. Since $b\neq 0$, this simplifies to $2a=a^2+b$ or $b=-a^2+2a$.

Thus, if $b\neq -a^2+2a$, the point $\left(a,b\right)$ cannot be on L_3, and we have $\left(p,q,r\right)=\underline{\mathbf{\left(1,-1,2\right)}}$.

Team Round - continued

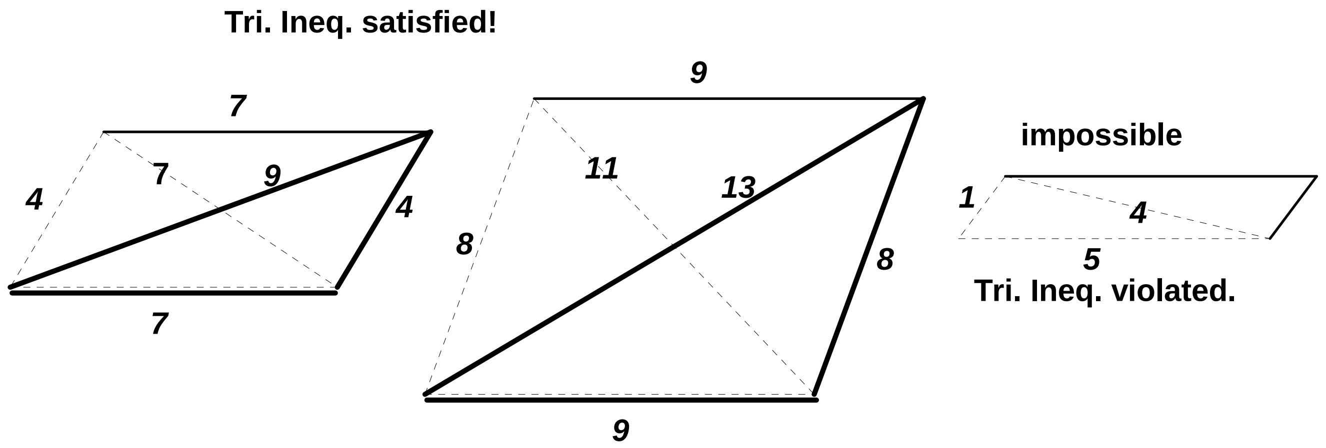
E) In any parallelogram, the sum of the squares of the lengths of the sides equals the sum of the squares of the lengths of its diagonals.

$2\left(x^2+(x+c)^2\right)=\left(x+3\right)^2+\left(x+5\right)^2 \Rightarrow 2\left(x+c\right)^2=6x+9+10x+25=16x+34$

$\left(x+c\right)^2=8x+17 \Rightarrow x^2+(2c-8)x+\left(c^2-17\right)=0$

$\Rightarrow x=\dfrac{\left(8-2c\right)\pm\sqrt{\left(2c-8\right)^2-4(1)\left(c^2-17\right)}}{2}=\dfrac{\left(8-2c\right)\pm\sqrt{4(33-8c)}}{2}=\left(4-c\right)\pm\sqrt{(33-8c)}$

Since x must be an integer, $(33-8c)$ must be a perfect square.
The only possibilities are 1, 2, 3 and 4.
$c = 1 \Rightarrow (33-8c) = 25$ and $x = 3 \pm 5 = 8$ and other side 9, diagonals 11 and 13 (ok)
$c = 3 \Rightarrow (33-8c) = 9$ and $x = 1 \pm 3 = 4$ and other side 7, diagonals 7 and 9 (ok)
$c = 4 \Rightarrow (33-8c) = 1$ and $x = 0 \pm 1 = 1$ and other side 5, diagonals 4 and 6 (rejected)
Therefore, there are two possible perimeters, 2(8 + 9) = **34** and 2(4 + 7) = **22**.
See diagram below.



Note:
$2\left(4^2+7^2\right)=7^2+9^2=130$
$2\left(8^2+9^2\right)=11^2+13^2=290$

MASSACHUSETTS MATHEMATICS LEAGUE
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Team Round - continued

E) To generate similar problems: Parallelogram w/sides: $x, x + c$ and diagonals: $x + a, x + b$

$$2\left(x^2+(x+c)^2\right)=\left(x+a\right)^2+\left(x+b\right)^2$$

$$\Rightarrow (x+c)^2 = x^2+2cx+c^2=\left(ax+bx\right)+\left(\frac{a^2+b^2}{2}\right)\Rightarrow x^2+\left(2c-a-b\right)x+\left(c^2-\frac{a^2+b^2}{2}\right)=0$$

$$\Rightarrow x=\frac{a+b-2c\pm\sqrt{\left(2c-a-b\right)^2+4\left(\frac{a^2+b^2}{2}-c^2\right)}}{2}$$

$$\Rightarrow x=\frac{a+b-2c\pm\sqrt{\left(2c-a-b\right)^2+2\left(a^2+b^2\right)-4c^2}}{2}$$

Simplifying the discriminant,

$$\left(2c-a-b\right)^2+2\left(a^2+b^2\right)-4c^2 = 4c^2+a^2+b^2-4ac-4bc+2ab+2a^2+2b^2-4c^2$$

$$= (a+b)^2+2\left(a^2+b^2\right)-4c(a+b)$$

Try (a, b)

$= (2, 4) \Rightarrow 36 + 2(20) – 24c = 76 – 24c = 4(19 – 6c) \Rightarrow c = 1$ only

$= (3, 6) \Rightarrow 81 + 2(45) – 36c = 171 – 36c = 9(19 – 4c) \Rightarrow$ none

$= (4, 7) \Rightarrow 121 + 2(65) – 44c = 251 – 44c \Rightarrow$ none

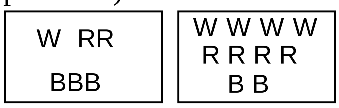
$= (3, 5) \Rightarrow 64 + 2(34) – 32c = 132 – 32c = 4(33 – 8c) \Rightarrow c = 1, 3, 4$ Bingo!

$= (3, 7) \Rightarrow 100 + 2(58) – 40c = 216 – 40c = 4(54 – 10c) \Rightarrow c = 5$ only

$= (5, 9) \Rightarrow 196 + 2(106) – 56c = 408 – 56c = 8(51 – 7c) \Rightarrow c = 7$ only

F) There are only two options for the draw from urn #1 – either *RW* or *RR*
(There were only 6 options: *RB* would leave 1 white and 1 red, *BB* would leave 1 white and 1 blue, *BW* would leave 2 reds and 2 blues and *WW* was impossible.)

$$P(RW)=\frac{2}{\binom{6}{2}}=\frac{2}{15},\ P(RR)=\frac{1}{\binom{6}{2}}=\frac{1}{15}$$



#1 #2

Note: $P(WW)=\frac{0}{15}, P(RB)=\frac{2\cdot 3}{15}=\frac{6}{15}, P(BB)=\frac{3}{15}, P(BW)=\frac{3}{15}$ and the sum of these probabilities is $\frac{15}{15}=1$. *RW* from #1 $\Rightarrow$ 5*W* 5*R* 2*B* in #2 *RR* from #1 $\Rightarrow$ 4*W* 6*R* 2*B* in #2

Now

$$P(\text{same color from \#2}) = P(WW \mid RW \text{ from \#1})+P(RR \mid RW \text{ from \#1})+P(BB \mid RW \text{ from \#1})$$
$$+P(WW \mid RR \text{ from \#1})+P(RR \mid RR \text{ from \#1})+P(BB \mid RR \text{ from \#1})$$

$$=\frac{2}{15}\left(\frac{\binom{5}{2}+\binom{5}{2}+\binom{2}{2}}{\binom{12}{2}}\right)+\frac{1}{15}\left(\frac{\binom{4}{2}+\binom{6}{2}+\binom{2}{2}}{\binom{12}{2}}\right)=\frac{2}{15}\cdot\frac{21}{66}+\frac{1}{15}\cdot\frac{22}{66}=\frac{64}{15\cdot 66}=\mathbf{\frac{32}{495}}$$

Addenda:
1B – (Thanks to Sung Ahn and Sam Solomon - Canton)
An interesting alternative approach:

By solving the 3rd equation for *z* and substituting for *z* in the 1st equation, we see that *y* must be $\frac{11}{12}$.

$$\begin{matrix} 14x+3y-7z=8 & & 14x-7z=k_1 \\ -8x+5y+4z=c & \text{becomes} & -8x+4z=k_2 \\ -2x+3y+z=2 & & -2x+z=k_3 \end{matrix}$$
, where each *k* is just a constant, namely,

$k_1=8-3y, k_2=c-5y$, and $k_3=2-3y$

Divide the equations through by 7, –4 and –1 respectively.

We have
$$\begin{aligned} 2x-z&=k_1/7 \\ 2x-z&=-k_2/4 \\ 2x-z&=-k_3 \end{aligned}$$

If the constants on the right hand side are all different, we have 3 parallel lines, which would produce no solutions.
If exactly two of the constants were equal, we would have 2 parallel lines (again no solution).
The only way the system could produce an infinite number of solutions is if all three constants are the same.

We require that $\frac{8-3y}{7}=\frac{5y-c}{4}=3y-2$.

For $y=\frac{11}{12}$, the first and last expressions evaluate to $\frac{3}{4}$.

$$\frac{5y-c}{4}=\frac{3}{4} \Rightarrow c=5y-3=5\left(\frac{11}{12}\right)-3=\frac{55-36}{12}=\frac{19}{12}$$.

Team D has been rewritten to request the parameters of the required conditions as an ordered triple.

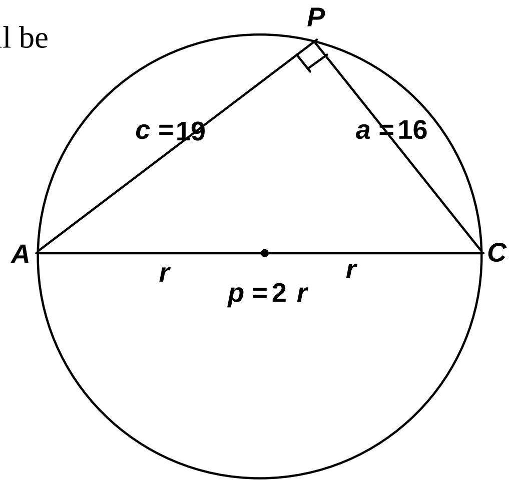
The listed answer to the original question was incorrect and the requested conditions were much more involved than anticipated.

The original question was:

For nonzero real constants *a* and *b*, the linear equations $y=ax+b$ and $\frac{x}{a}-\frac{y}{b}=1$ intersect along $y=x$. Compute all ordered pairs (a,b) for which this is not possible.

The original problem 5C intended *P* to be in the interior of the circle, but did not specify that restriction.

Certainly, if $B = P = D$ and $m\angle APC = 90$, then $\angle APC$ will be inscribed in a semi-circle and $r = \frac{AC}{2}$ and $AC = \sqrt{16^2 + 19^2} = \sqrt{617}$.



Comparing the squares, $\left(\frac{5}{2}\sqrt{17}\right)^2 = \frac{25\cdot 17}{4} = \frac{425}{4}$, $\left(\frac{\sqrt{617}}{2}\right)^2 = \frac{617}{4}$

and we have an answer larger than the "official" answer.
Both answers were accepted.

Extension: Suppose point *P* had been specified to be on the circle, but the perpendicular condition were omitted. Could we draw any conclusions about the radius *r*?
By making the angle at *P* acute (or obtuse), what happens to *r*?
Consider the radius of a circle circumscribed about a triangle. Let *K* denote the area of the triangle.
Knowing that $r_{cc} = \frac{abc}{4K}$ and $K = \frac{1}{2}ab\sin\theta$, where θ is the included angle, we have

$r_{cc} = \frac{16\cdot 19\cdot p}{4\left(\frac{1}{2}\cdot 16\cdot 19\cdot \sin P\right)} = \frac{p}{2\sin P} = \boxed{\frac{p}{2}\cdot\frac{1}{\sin P}}$ which reduces to the above answer for $P = 90°$. But,

for any inscribed angle, $\frac{1}{\sin P} \geq 1$. The problem is that *P* and *p* are not independent of each other.
Using the Law of Cosines and the fact that $\sin^2\theta + \cos^2\theta = 1$, for angle θ, we have

$p^2 = 16^2 + 19^2 - 2\cdot 16\cdot 19\cos P \Rightarrow \cos P = \frac{617 - p^2}{2\cdot 16\cdot 19}$ and $\sin^2 P = 1 - \left(\frac{617 - p^2}{2\cdot 16\cdot 19}\right)^2$. Consequently,

substituting in the boxed expression above, $r_{cc} = \frac{p}{2\sqrt{1 - \left(\frac{617 - p^2}{2\cdot 16\cdot 19}\right)^2}}$. Constructing a lookup table for

integer values of *c* from 4 to 34 is enlightening. Mr. TI gives these results:

p	4	5	…	9	10	11	…	24	25
R	13.22	10.97		9.53	9.50	9.51		12.03	12.50
p	26	27	28	29	30	31	32	33	34
R	13.06	13.74	14.56	15.60	16.95	18.80	21.54	26.18	36.74

As *P* becomes very acute $(P \to 0 \Rightarrow p \to 3)$ and as *P* becomes very obtuse $(P \to 180° \Rightarrow p \to 35)$, as expected from the Triangle Inequality. In either case, *r* becomes longer and longer $(r \to \infty)$.
Without the perpendicular restriction, there is no maximum value of *r*.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2015 ANSWERS

Round 1 Alg 2: Simultaneous Equations and Determinants

A) -14 B) $\frac{19}{12}$ C) 1, 2 , 4, 7

Round 2 Alg 1: Exponents and Radicals

A) 4 B) 3721 C) $(-4,-1),\ (-8,-7)$

Round 3 Trigonometry: Anything

A) 420 B) $\frac{5\pi}{6} < x < \frac{7\pi}{6}$ or $\left(\frac{5\pi}{6},\frac{7\pi}{6}\right)$ C) 16

Round 4 Alg 1: Anything

A) 65 B) $\left(-\frac{1}{2},2,4\right)$ C) 6

Round 5 Plane Geometry: Anything

A) 140 B) 12 C) $\frac{5}{2}\sqrt{17}$

Round 6 Alg 2: Probability and the Binomial Theorem

A) $\frac{15}{32}$ B) -6 C) $\left(\frac{1}{16},\frac{5}{128}\right)$

Team Round

A) -4 D) $(1,-1,2)$

B) -1 and 19 E) 22 and 34

C) $\sqrt{13}$ F) $\frac{32}{495}$