MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2015
ROUND 1 COMPLEX NUMBERS (No Trig)

ANSWERS

A) ________________________________

B) ________________________________

C) (_____________ , _____________)

A) $i^p = -i$ for some *prime* p. Compute the minimum value of $p > 100$.

B) It is easy to verify that $(1+i)^4 = -4$. [$(1+i)^4 = \left((1+i)^2\right)^2 = (2i)^2 = -4$]

For exactly 6 integers k between 1 and 25, the value of $(1+i)^k$ is a real number.

Compute the sum of these 6 powers of $(1+i)$.

C) Let $9i$ be added to the sum of the four 4^{th} roots of 16. If A denotes this sum and $A^3 = a + bi$, compute the ordered pair (a,b).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2015 SOLUTION KEY

Round 1

A) Since $i^3 = -i$ and $i^4 = 1$, it follows that $i^3 = i^7 = i^{11} = ... = -i$
The prime we seek is 3 more than a multiple of 4.
The first value we check is 103.
To verify primeness, we test for divisibility by 2, 3, 5 and 7.
Any composite number *N* must have a factor which is less than or equal to $\sqrt{N}$.
All four possible factors fail and **<u>103</u>** is prime.

B) The 6 values of *k* are the multiples of 4, namely 4,8,12,16,20 and 24.
The 6 real numbers are $-4, 16, -64, 256, -1024, \text{ and } 4096$, which produce a sum of **<u>3276</u>**.

C) $x^4 = 16 \Leftrightarrow x^4 - 16 = \left(x^2 - 4\right)\left(x^2 + 4\right) = 0$
The four 4th roots of 16 are $\pm 2, \pm 2i$; so their sum will be 0.
$(0 + 9i)^3 = 729i^3 = -729i$. Therefore, $(a,b) = \underline{\mathbf{(0,-729)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2015
ROUND 2 ALGEBRA 1: ANYTHING

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) There are 36 marbles in a bag. 16 are red, 7 are white and the remaining marbles are blue. The blue marbles are removed, each one being replaced by either a red or a white marble. How many blue marbles must be replaced by red marbles so that the ratio of red to white marbles is 2 : 1?

B) For integers x and y, $3x+8y=101$, $x>0$ and, $y>0$.
Compute all possible sums of $x + y$.

C) Sara, running in a 5-mile race, ran the first 2 miles in 15 minutes. Her overall average speed for the entire race was 25% more than her average speed for the first 2 miles. Compute her average speed for the last 3 miles (in miles per hour).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2015 SOLUTION KEY

Round 2

A) Let x denote the # of blue marbles replaced by a red marble.
Initially, $(R,W)=(16,7)$. After replacement, $R:W=2:1$.
$16+x=2(7+(13-x))=40-2x \Rightarrow 3x=24 \Rightarrow x=\underline{\mathbf{8}}$.

B) $3x+8y=101 \Rightarrow x=\dfrac{101-8y}{3}=33-2y+2\left(\dfrac{1-y}{3}\right)$
The slope of this line is $\dfrac{-3}{8}$ or $\dfrac{3}{-8}$ and clearly, for y = 1, the value of x will be an integer, namely, x = 33 – 2 + 2(0) = 31.
Increasing y by 3 will decrease x by 8, changing the sum by –5.
Thus, (31, 1) ⇒ **<u>32</u>** (23, 4) ⇒ **<u>27</u>** (15,7) ⇒ **<u>22</u>** (7, 10) ⇒ **<u>17</u>**
For any other ordered pairs, either x or y is negative, so these are the only possible ordered pairs.

C) For the first 2 miles Sara's rate was $\dfrac{2}{\frac{15}{60}}=8$ mph.
Solution #1: If her overall average rate was 25% faster than her average rate for the first 2 miles, then she averaged $\dfrac{5}{4}\cdot 8=10$ mph for the 5 mile race.
Thus, 2 miles @ 8 mph plus 3 miles @ x mph = 5 miles at 10 mph.
$\dfrac{2}{8}+\dfrac{3}{x}=\dfrac{5}{10} \Leftrightarrow \dfrac{3}{x}=\dfrac{1}{4} \Rightarrow x=\underline{\mathbf{12}}$
Solution #2: Let the time to complete the last 3 miles be A minutes.
$R=\dfrac{D}{T}=\dfrac{5}{\frac{15+A}{60}}=8\left(\dfrac{5}{4}\right)=10 \Leftrightarrow \dfrac{300}{15+A}=10 \Rightarrow A=15 \Rightarrow$ 5 min per mile ⇔ **<u>12</u>** mph

Solution #3: $8\cdot\dfrac{3}{2}=\underline{\mathbf{12}}$ HUH??!

Rate · Time = Distance ⇒ $R=\dfrac{D}{T}$ Let r and x denote the rates for the 2 mile and 3 mile legs respectively. Then: $R=\dfrac{5}{\frac{2}{r}+\frac{3}{x}}=\dfrac{5}{4}r \Rightarrow \dfrac{1}{\frac{2x+3r}{rx}}=\dfrac{r}{4} \Rightarrow \dfrac{\cancel{r}x}{2x+3r}=\dfrac{\cancel{r}}{4}$
Cross multiplying, $4x=2x+3r \Rightarrow x=\dfrac{3}{2}r$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2015
ROUND 3 PLANE GEOMETRY: AREAS OF RECTILINEAR FIGURES

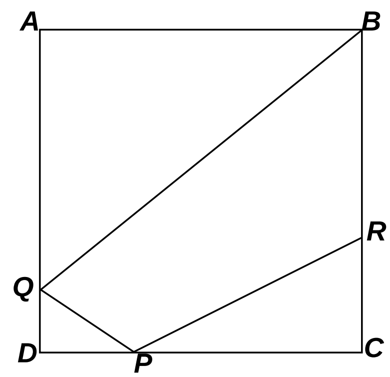
ANSWERS

A) ______________________________

B) ______________________________

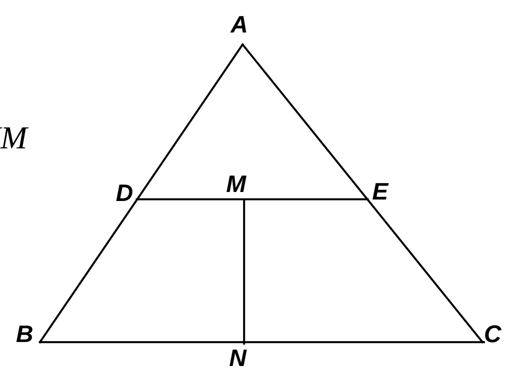
C) ______________________________

A) Given: *ABCD* is a square, $QP = RC = 5$, $DP = 4$, $PR = 13$
Compute the area of quadrilateral *BRPQ*.



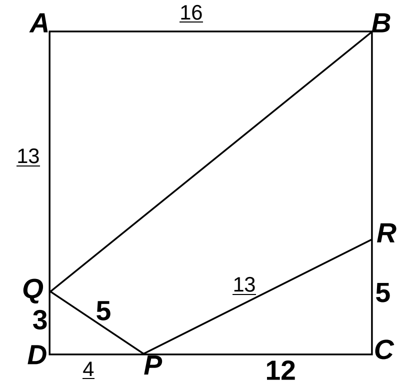
B) In square *ABCD*, *M* is the midpoint of $\overline{BC}$ and *T* is a trisection point of $\overline{AB}$.
Compute the largest possible area of ΔTDM, if $AD = 6$.

C) ΔABC is equilateral with side 6. $\overline{DE} \parallel \overline{BC}$
M and *N* are midpoints of $\overline{DE}$ and $\overline{BC}$, respectively.
If the areas of ΔADE, trapezoid *DBNM* and trapezoid *ECNM* are equal, compute *MN*.



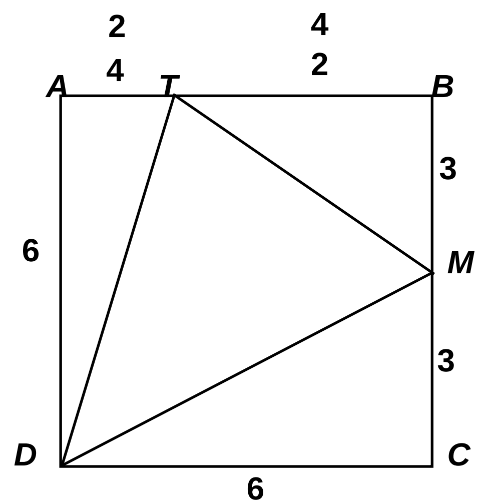
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2015 SOLUTION KEY

Round 3



A) $DP = 4, PR = 13, AB = 16, QA = 13$

$\Rightarrow area(BRPQ) = 16^2 - (6 + 30 + 104) = \underline{\mathbf{116}}$.

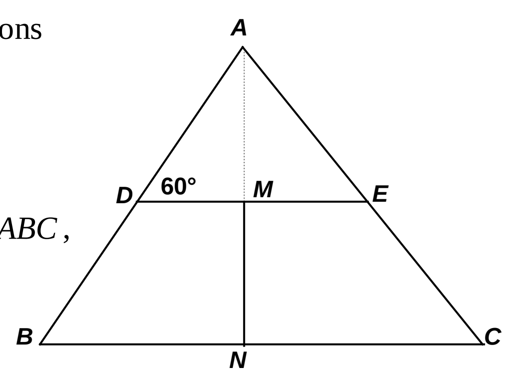


B) $BM = CM = 3$

If $AT = 2,\ BT = 4$, then $area(TMD) = 36 - (6 + 6 + 9) = 15$.

If $AT = 4,\ BT = 2$, then $area(TMD) = 36 - (12 + 3 + 9) = 12$.

Thus, the largest area is $\underline{\mathbf{15}}$.



C) Since ΔABC is equilateral and $AB = 6$,

the area of ΔABC is $\dfrac{6^2\sqrt{3}}{4} = 9\sqrt{3}$, so each of the three regions

has area $3\sqrt{3}$. The area of ΔADE is

$\dfrac{AD^2\sqrt{3}}{4} = 3\sqrt{3} \Rightarrow AD = 2\sqrt{3} \Rightarrow DM = \sqrt{3},\ AM = 3$.

A, M and N are collinear and, as an altitude in equilateral ΔABC,

$AN = 3\sqrt{3}$.

Thus, $MN =$ $\underline{\mathbf{3\sqrt{3} - 3}}$ or $\underline{\mathbf{3\left(\sqrt{3} - 1\right)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2015
ROUND 4 ALG 1: FACTORING AND ITS APPLICATIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Factor over the integers: $182a - 12a^2 - 2a^3$

B) Compute all values of x over the reals for which $x^4 - (13x - 30)^2 = 0$.

C) Factor over the integers: $x^2 + 4x - 16y^2 + 16y$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2015 SOLUTION KEY

Round 4

A) $182a-12a^2-2a^3=2a\left(91-6a-a^2\right)=\underline{\mathbf{2a(13+a)(7-a)}}$, $\underline{\mathbf{-2a(a+13)(a-7)}}$ or equivalent.

B) As the difference of perfect squares,

$x^4-(13x-30)^2 \Leftrightarrow \left(x^2+13x-30\right)\left(x^2-13x+30\right)=(x+15)(x-2)(x-3)(x-10)$

Setting equal to zero, we have $x=\underline{\mathbf{-15,\ 2,\ 3,\ 10}}$ (in any order).

C) Completing the squares in both the x- and y-expressions, we have the difference of perfect squares. Note the "fudge factors", namely $+4$ and $-16\left(\frac{1}{4}\right)$ sum to zero, so the original polynomial has not been changed!!

$x^2+4x-16y^2+16y \Leftrightarrow \left(x^2+4x+4\right)-16\left(y^2-y+\frac{1}{4}\right)$

$\Leftrightarrow (x+2)^2-4^2\left(y-\frac{1}{2}\right)^2=(x+2)^2-(4y-2)^2$

$\Leftrightarrow (x+2+4y-2)(x+2-4y+2)=\underline{\mathbf{(x+4y)(x-4y+4)}}$

Alternately, grouping the quadratic terms and the linear terms, we have

$x^2+4x-16y^2+16y$

$\Leftrightarrow \left(x^2-16y^2\right)+4(x+4y)$

$\Leftrightarrow (x+4y)(x-4y)+4(x+4y)$

$\Leftrightarrow \underline{\mathbf{(x+4y)(x-4y+4)}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2015
ROUND 5 TRIG: FUNCTIONS OF SPECIAL ANGLES

ANSWERS

A) ________________________________

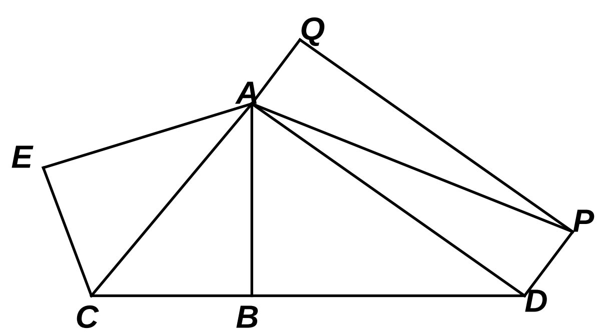
B) ________________________________

C) ________________________________

A) Compute: $\dfrac{\left(\sec 330° \cdot \sin 240° \cdot \tan 495°\right)^3}{\left(\csc 120° \cdot \cot 225°\right)^2}$

B) $P = \sin x \cdot \cos 2x \cdot \tan 3x \cdot \cot 4x \cdot \sec 5x \cdot \csc 6x$
Compute P, if $x = 15°$.

C) Given: $\overline{AB} \perp \overline{CD}$, $\overline{CA} \perp \overline{AD}$, $\overline{CE} \perp \overline{EA}$, $\overline{PD} \perp \overline{AD}$,
C, A, and Q are collinear, $m\measuredangle ECB = 120°$
$QA = PD = \sqrt{3}$, $AB = 6$, $BD = 6\sqrt{3}$,
Compute $AD + BC + EA + AP$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2015 SOLUTION KEY

Round 5

A) $$\frac{\left(\sec 330^\circ \cdot \sin 240^\circ \cdot \tan 495^\circ\right)^3}{\left(\csc 120^\circ \cdot \cot 225^\circ\right)^2} = \frac{\left(\frac{2}{\sqrt{3}} \cdot \frac{-\sqrt{3}}{2} \cdot -1\right)^3}{\left(\frac{2}{\sqrt{3}} \cdot 1\right)^2} = \frac{1}{\frac{4}{3}} = \underline{\mathbf{\frac{3}{4}}}\ .$$

B) $$\sin 15^\circ \cdot \cos 30^\circ \cdot \tan 45^\circ \cdot \cot 60^\circ \cdot \sec 75^\circ \cdot \csc 90^\circ = \sin 15^\circ \cdot \frac{\sqrt{3}}{2} \cdot 1 \cdot \frac{1}{\sqrt{3}} \cdot \frac{1}{\cos 75^\circ} \cdot 1$$

$$= \frac{1}{2}\frac{\sin 15^\circ}{\cos 75^\circ} = \frac{1}{2}\frac{\sin 15^\circ}{\sin 15^\circ} = \underline{\mathbf{\frac{1}{2}}}\ .$$

C) Since $\overline{AB}$ and $\overline{BD}$ are sides of a right triangle with lengths in a ratio of $1:\sqrt{3}$, ΔBAD is a 30-60-90 triangle. Similarly, ΔABC is a 30-60-90 triangle. Since $\measuredangle ECB = 120^\circ$, ΔEAC is also a 30-60-90 triangle.

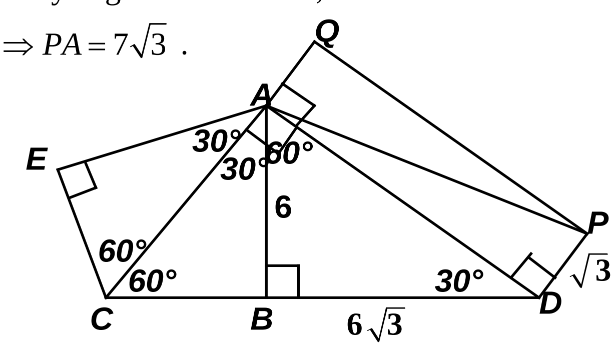
Thus, $AD = 12$, $BC = 2\sqrt{3}$, $AC = 4\sqrt{3} \Rightarrow EC = 2\sqrt{3}, EA = 6$

ΔPAD is not a special right triangle, but, applying the Pythagorean Theorem, we have the last length needed. $PA^2 = 12^2 + \left(\sqrt{3}\right)^2 = 147 = 49 \cdot 3 \Rightarrow PA = 7\sqrt{3}$.

Therefore,

$AD + BC + EA + AP = 12 + 2\sqrt{3} + 6 + 7\sqrt{3} =$

$\underline{\mathbf{18 + 9\sqrt{3}}}$ or $\underline{\mathbf{9\left(2 + \sqrt{3}\right)}}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2015
ROUND 6 PLANE GEOMETRY: ANGLES, TRIANGLES AND PARALLELS

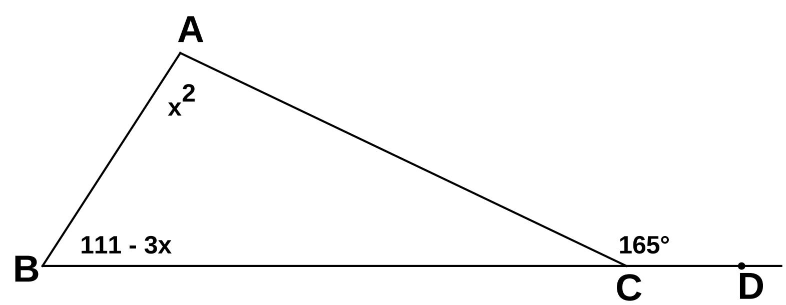
ANSWERS

A) ______________________________

B) ______________________________

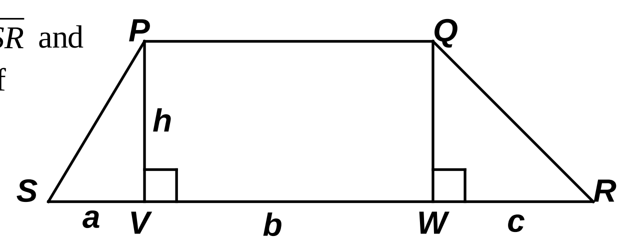
C) ______________________________

A) Compute the largest possible degree-measure of an angle of ΔABC.



B) An equilateral triangle *EDC* is constructed in the interior of square *ABCD*. $\overline{EF}$ is an altitude to $\overline{AB}$. Compute $m\angle ABE + m\angle FED$, in degrees.

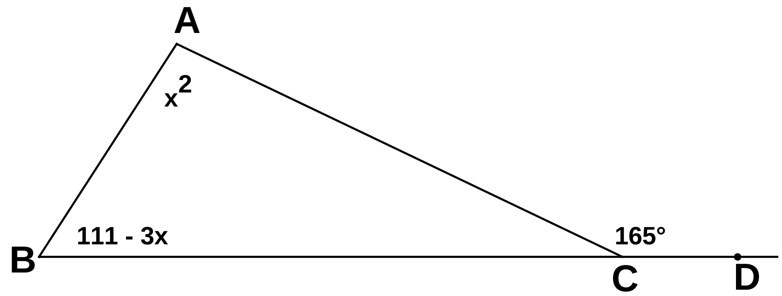
C) In trapezoid *PQRS*, $\overline{PQ} \parallel \overline{RS}$, $\overline{PV}, \overline{QW} \perp \overline{SR}$ and the ratio of the area of ΔPSV to the area of ΔQWR is $2:3$.
If $a+b+c=60$, $b:c=5:6$, and $h:b=9:40$,
compute the area of $\underline{\Delta PVR}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2015 SOLUTION KEY

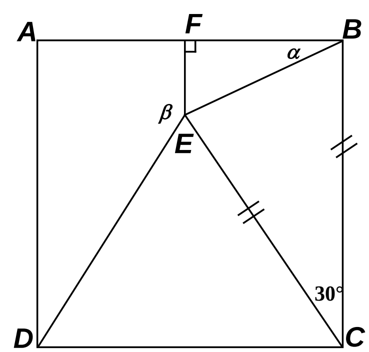
Round 6

A)

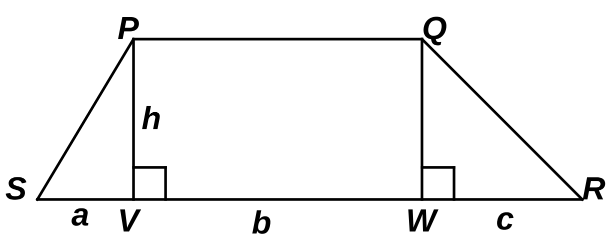


Since the measure of any exterior angle of a triangle is equal to the measure of the sum of the two interior angles, we have $x^2+(111-3x)=165 \Rightarrow x^2-3x-54=(x-9)(x+6)=0 \Rightarrow x=9,-6$
$x=9$ produces angles of 81, 84 and 15, but $x=-6$ produces angles of 36, 129 and 15.
Thus, the largest possible degree-measure is <u>**129**</u>.

B) Since $EC=CD$ and $BC=CD$, by transitivity, $BC=EC$ and ΔBEC is isosceles. $m\angle CBE=m\angle CEB=75° \Rightarrow \alpha=15°$
In trapezoid , $\beta=(360-2\cdot 90°-30°)=150°$.
Therefore, $\alpha+\beta=\underline{\mathbf{165}}°$.



C)



$$area(PSV):area(QWR)=\frac{1}{2}ah:\frac{1}{2}ch=2:3 \Rightarrow a:c=2:3$$

$$\begin{cases} b:c=5:6 \\ a:c=2:3 \end{cases} \Rightarrow a:b:c=4:5:6$$

Thus, $a+b+c=4n+5n+6n=60 \Rightarrow n=4 \Rightarrow (a,b,c)=(16,20,24)$

$h:b=9:40 \;\Rightarrow\; h=4.5$

Thus, the area of ΔPVR is $\frac{1}{2}4.5(20+24)=4.5(22)=\underline{\mathbf{99}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2015
ROUND 7 TEAM QUESTIONS
ANSWERS

A) ____________________________ D) (______ , ______ , ______)

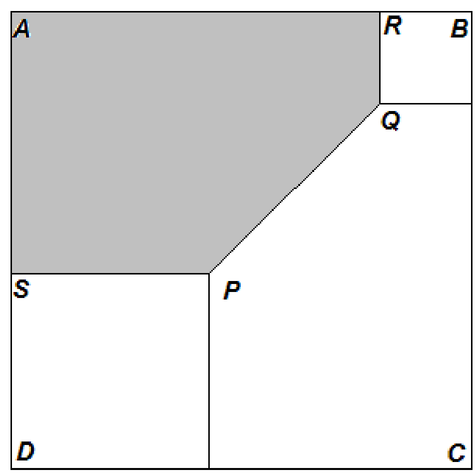
B) ____________________________ E) (__________ , __________)

C) ____________________________ F) ____________________________

A) Determine the integer value of n for which $(1+i)^{11}+(1-i)^{n}=-16+16i$.

B) One million lottery tickets are numbered 000000 through 999999.
Let A be the set of lucky lottery tickets.
A lucky lottery ticket has the form *abcxyz*, where $a+b+c=x+y+z$.
Let B be the set of unlucky lottery tickets.
An unlucky lottery ticket is defined to be one where the 6 digits sum to 27.
For 1) – 5) below, list the numbers of the true statements.
Note $N(X)$ denotes the number of elements in set X.

1) $N(B)>N(A)$
2) $N(B)<N(A)$
3) $N(B)=N(A)$
4) With respect to the given definitions, no ticket is both lucky and unlucky.
5) With respect to the given definitions, at least one ticket is both lucky and unlucky.



C) In square *ABCD*, the squares in opposite corners have sides 1 and 2, respectively. A square with side *PQ* has an area equal to sum of the areas of the small squares.
Compute the area of the shaded region.

D) Given: $a^2+b^2+ab=12$, $b^2+c^2+bc=13$, $a^2+c^2+ac=19$
If $a>0$, compute (a,b,c) over the rational numbers.

E) The value of the expression $N=\sin\left(\frac{11\pi}{3}\right)+\cos(600^\circ)+\sin^2\left(\frac{9\pi}{4}\right)-\tan^2(495^\circ)-3\tan(540^\circ)$
satisfies the inequality $a<N<b$, where a and b are integers and $b-a=1$.
Compute the ordered pair (a, b).

F) ΔABC is known to be isosceles, but it is not known which angle is the vertex angle.
$\overrightarrow{BP}$ is a trisector of $\angle B$, so that $m\angle PBC<m\angle ABP$ (P is on $\overline{AC}$).
$\overrightarrow{CQ}$ is a bisector of $\angle C$ (Q is on $\overline{AB}$).
$\overrightarrow{BP}\cap\overrightarrow{CQ}=\{D\}$. $m\angle BDC=140^\circ$. Compute all possible $m\angle A$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2015 SOLUTION KEY

Team Round

A) $(1+i)^2 = 2i \Rightarrow (1+i)^{11} = (2i)^5(1+i) = 32\cancel{i^4}i(1+i) = -32+32i$

Thus, we require that $(1-i)^n = -(-32+32i)+(-16+16i) = 16-16i$.

$(1-i)^2 = -2i \Rightarrow (1-i)^8 = (-2i)^4 = 16i^4 = 16$

Therefore, $(1-i)^9 = 16(1-i) = 16-16i \Rightarrow \quad n = \underline{\mathbf{9}}$.

Alternate Solution:

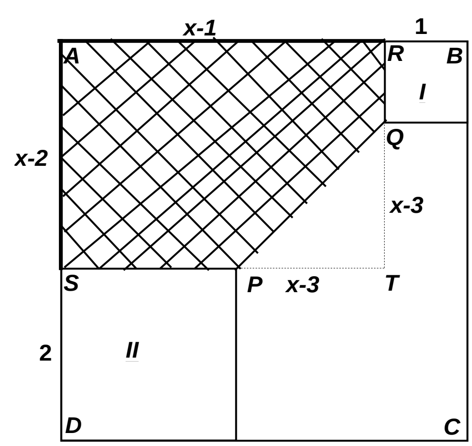
Using polar (or cis form), $(1-i)^n = 16-16i \Leftrightarrow \left(\sqrt{2},-45°\right)^n = \left(16\sqrt{2},-45°\right)$

$\Rightarrow\ 2^{n/2} = 2^{4.5} \Rightarrow n = \underline{\mathbf{9}}$.

B) Suppose $L = abcxyz$ is a lucky lottery ticket.
Consider the companion lottery ticket M with the number $(9-a)(9-b)(9-c)xyz$.
Since $a+b+c = x+y+z$, we have
$(9-a)+(9-b)+(9-c)+x+y+z = 27-(a+b+c)+(x+y+z) = 27$
Thus, each companion lottery ticket's digits total 27.
For each ticket L, there is exactly one ticket M and vice versa.
Because of this one-to-one correspondence, we see there are as many lucky lottery tickets as there are these companion lottery tickets. Amazingly, $N(A) = N(B)$ and (3) is true.
If a lottery ticket is lucky then $a+b+c = x+y+z$ and, regardless of whether these sums are even or odd, the sum of the 6 digits will be even; hence, never equal to one of the companion lottery tickets.
Since (4) is true, (5) must be false. Thus, (**<u>3) and (4)</u>** are true.

C) Let x be the side of square $ABCD$.
D, P, Q and B are collinear, so
$BD = 2\sqrt{2}+PQ+\sqrt{2} = x\sqrt{2}$
$\Rightarrow PQ = (x-3)\sqrt{2}$
Area(I) + Area(II) = Area(Square on $\overline{PQ}$) $\Rightarrow$

$PQ^2 = 1^2+2^2 = 5$ and we have $\sqrt{5} = (x-3)\sqrt{2} \Rightarrow x = 3+\dfrac{\sqrt{10}}{2}$.



Thus, area of the shaded region is equal to the area of rectangle $ARTS$ minus the area of triangle PQT.

$$\left(1+\frac{\sqrt{10}}{2}\right)\left(2+\frac{\sqrt{10}}{2}\right)-\frac{1}{2}\cdot\left(\frac{\sqrt{10}}{2}\right)^2 = 2+\frac{3}{2}\sqrt{10}+\frac{5}{2}-\frac{5}{4} = \underline{\mathbf{\frac{13+6\sqrt{10}}{4}}}$$

Team Round - continued

D) $\begin{cases} (1)\ a^2+b^2+ab=12 \\ (2)\ b^2+c^2+bc=13 \\ (3)\ a^2+c^2+ac=19 \end{cases}$ Subtracting $(2)-(1) \Rightarrow 1 = c^2-a^2+b(c-a)=(c-a)(c+a+b)$

Similarly, $(3)-(2) \Rightarrow\ 6=(a-b)(a+b+c)$

By transitivity, $\dfrac{1}{c-a}=\dfrac{6}{a-b} \Rightarrow \boxed{b=7a-6c}$ (4).

Substituting in (1),

$a^2+(7a-6c)^2+a(7a-6c)=12 \Leftrightarrow a^2+\left(49a^2-84ac+36c^2\right)+7a^2-6ac=12$

$\Leftrightarrow\ 57a^2-90ac+36c^2=12\ \Leftrightarrow\ \boxed{19a^2-30ac+12c^2=4}$

Substituting in (2),

$(7a-6c)^2+c^2+(7a-6c)c=13 \Leftrightarrow 49a^2-84a+36c^2+c^2+7ac-6c^2$

$\Leftrightarrow\ \boxed{49a^2-77ac+31c^2=13}$

Factoring these trinomials would be fruitless, unless they were equal to zero!
Multiplying the first equation by 13 and the second by 4, we get our wish.

$13\left(19a^2-30ac+12c^2=4\right)+4\left(49a^2-77ac+31c^2\right)=51a^2-82ac+32c^2=0$

$\Leftrightarrow\ (3a-2c)(17a-16c)=0 \Rightarrow c=\dfrac{3}{2}a,\ \dfrac{17}{16}a$

Substituting in (4), $b=7a-6\left(\dfrac{3}{2}a\right)=-2a$

Substituting in (1), $a^2+(-2a)^2+a(-2a)=3a^2=12$ and $a>0 \Rightarrow a=2 \Rightarrow \underline{\mathbf{(2,-4,3)}}$.

Alternately, subtracting (2) from (1) and factoring, we have $(a-c)(a+b+c)=-1$. Using (4), $(a-c)(8a-5c)=-1$. For *integer* solutions, one factor would be 1 and the other would be -1.

$\begin{cases} a-c=-1 \\ 8a-5c=1 \end{cases} \Rightarrow (a,b,c)=\underline{\mathbf{(2,-4,3)}}$, but $\begin{cases} a-c=1 \\ 8a-5c=-1 \end{cases} \Rightarrow (a,b,c)=(-2,4,-3)$, rejected since $a<0$.

FYI:
The other substitution for c produces *irrational* solutions.

$b=7a-6\left(\dfrac{17}{16}a\right)=\dfrac{5}{8}a \Rightarrow\ a^2+\left(\dfrac{5}{8}a\right)^2+a\left(\dfrac{5}{8}a\right)=\dfrac{129}{64}a^2=12 \Rightarrow\ a^2=\dfrac{4(64)}{43}$ and $a>0 \Rightarrow a=\dfrac{16}{\sqrt{43}}$

$\Rightarrow\ (a,b,c)=\left(\dfrac{16}{\sqrt{43}},\dfrac{10}{\sqrt{43}},\dfrac{17}{\sqrt{43}}\right)$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2015 SOLUTION KEY

Team Round - continued

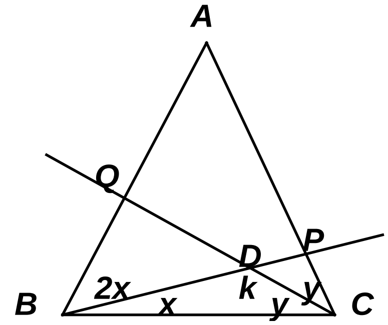
E) $\sin\left(\frac{5\pi}{3}\right)+\cos(240°)+\sin^2\left(\frac{\pi}{4}\right)-\tan^2(135°)-3\tan(180°)=-\frac{\sqrt{3}}{2}-\frac{1}{2}+\frac{1}{2}-1-0=-\frac{\left(\sqrt{3}+2\right)}{2}$

Since $\sqrt{1}<\sqrt{3}<\sqrt{4}$, we know $1<\sqrt{3}<2 \Leftrightarrow \frac{1}{2}<\frac{\sqrt{3}}{2}<1$

Adding 1, we have $\frac{3}{2}<\frac{\sqrt{3}+2}{2}<2$ and, therefore, (a, b) = **<u>(–2, –1)</u>**.

Some students remember that $\sqrt{3}\approx 1.732$ (the year of George Washington's birth) and, therefore, $\frac{\left(\sqrt{3}+2\right)}{2}\approx\frac{3.732}{2}\approx 1.8^{+}$ and the same result follows.

F) $\overrightarrow{BP}$ is a <u>trisector</u> of $\angle B$, so that $m\angle PBC < m\angle ABP$.
$\overrightarrow{CQ}$ is a <u>bisector</u> of $\angle C$.
$m\angle BDC = k = 140°$



Case 1: A is the vertex angle

$$\begin{cases} x+y=40 \\ 3x=2y \end{cases} \Rightarrow (x,y)=(16,24) \text{ and } m\angle A = 180-(2\cdot 48)=\underline{\mathbf{84°}}$$

Case 2: B is the vertex angle (No solution)

$$\begin{cases} x+y=40 \\ m\measuredangle A = 2y = 180-(3x+2y) \end{cases}$$

$$\Rightarrow 3x+4y=180 \Rightarrow 3x+4(40-x)=180 \Rightarrow 160-x=180 \Rightarrow x=-20$$

Case 3: C is the vertex angle

$$\begin{cases} x+y=40 \\ m\measuredangle A = 3x = 180-(3x+2y) \end{cases}$$

$$\Rightarrow 6x+2y=180 \Rightarrow 3x+(40-x)=90 \Rightarrow (x,y)=(25,15) \Rightarrow m\angle A = \underline{75°}$$

Additional Challenges:
Suppose that $m\angle BDC = k°$.

- Show that if $k=130$, ΔABC is equilateral.
- Show that if A is the vertex angle, $105<k<180$.
- Show that if $k=125°$, there are 3 possible measures for $\angle A$, namely $30°, 48°, 52.5°$.
- Is there a k-value which gives three different <u>integer</u> values for $m\angle A$?

Your analysis can start here:

$$\begin{cases} x+y=180-k \\ 3x=2y \end{cases} \Rightarrow x=72-\frac{2k}{5},\ y=108-\frac{3k}{5}.\ A \text{ as vertex angle gives } A=\frac{12}{5}k-252,\ B=C=216-\frac{6}{5}k$$

Talk these questions over with your teammates.
Share your ideas with your coach and/or me (olson.re@gmail.com). Thanks.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2015 ANSWERS

Round 1 Algebra 2: Complex Numbers (No Trig)

A) 103 B) 3276 C) $(0,-729)$

Round 2 Algebra 1: Anything

A) 8 B) 32, 27, 22, 17 C) 12

Round 3 Plane Geometry: Area of Rectilinear Figures

A) 116 B) 15 C) $3\left(\sqrt{3}-1\right)$ or equivalent

Round 4 Algebra: Factoring and its Applications

A) $2a(13+a)(7-a)$ or $-2a(a+13)(a-7)$ B) $-15, 2, 3, 10$ C) $(x+4y)(x-4y+4)$

Note: Any order of the given factors or roots is allowed.

Round 5 Trig: Functions of Special Angles

A) $\frac{3}{4}$ B) $\frac{1}{2}$ C) $18+9\sqrt{3}$ or $9\left(2+\sqrt{3}\right)$

Round 6 Plane Geometry: Angles, Triangles and Parallels

A) 129 B) 165 C) 99

Team Round

A) 9

B) 3, 4

C) $\frac{13+6\sqrt{10}}{4}$

D) $(2,-4,3)$

E) $(-2,-1)$

F) 75°, 84°