MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2015
ROUND 1 TRIG: RIGHT ANGLE PROBLEMS, LAWS OF SINES AND COSINES

ANSWERS

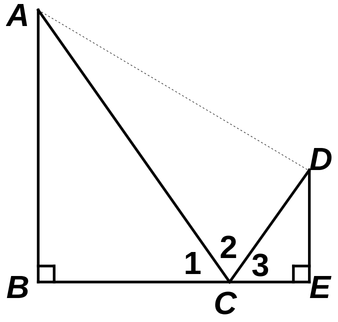
A) (_____________ , _____________)

B) ________________________________

C) (_____________ , _____________)

A) Right triangle *ABC* has sides of length $(141, b, c)$, where 141 is the length of the short leg and b is the length of the long leg. If *ABC* is similar to ΔDEF , whose sides have integer lengths and whose perimeter is 12. Determine the ordered pair (b, c).

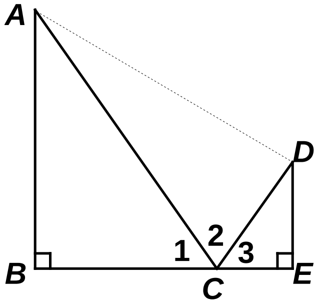
B) ΔABC and ΔCDE are right triangles, where *B*, *C* and *E* are collinear, $BE = 9$ and $BC = CE + 4$.
If $m\angle 1 = m\angle 2 = m\angle 3$, compute *AD*.



C) In ΔABC, $AC = 20$, $\sin C = \frac{\sqrt{7}}{4}$, $m\angle A = 2 \cdot m\angle C$, and $\cos B = \cos^2 C$.
The area of ΔABC in simplest form is $K\sqrt{L}$.
Determine the ordered pair (K, L).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2015 SOLUTION KEY

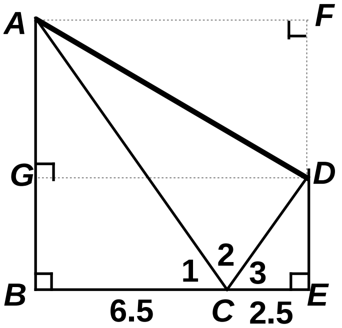
Round 1



A) ΔDEF must be a 3-4-5 triangle. Since $141 = 47\cdot 3$,
$(b,c) = (47\cdot 4, 47\cdot 5) = \underline{\mathbf{(188,\ 235)}}$.

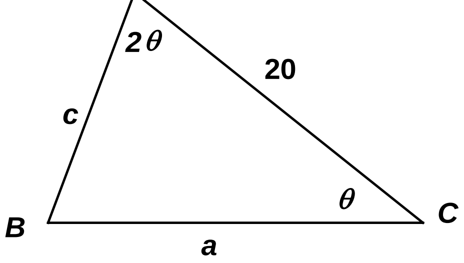
B) $m\angle 1 = m\angle 2 = m\angle 3 = 60°$,
$BC + CE = 2CE + 4 = 9 \Rightarrow CE = 2.5, BC = 6.5$ Both of these sides are opposite a 30° angle in a 30-60-90 right triangle. Thus, the hypotenuses are 5 and 13. Applying the Law of Cosines to ΔACD,

$$AD^2 = 5^2 + 13^2 - 2\cdot 5\cdot 13\cos 60° = 25 + 169 - 130\cdot\frac{1}{2} = 194 - 65 = 129 \Rightarrow AD = \underline{\sqrt{\mathbf{129}}}.$$



Solution #2 (Norm Swanson – Hamilton Wenham – retired)
Construct $\overline{DG}$ and point F so that $\overline{DG} \perp \overline{AB}$, $\overline{AF} \perp \overline{FDE}$.
$AF = 9$, ΔDEC and ΔABC are 30-60-90 right triangles, $FE = AB$.
$FD = AB - GB = FE - DE = 6.5\sqrt{3} - 2.5\sqrt{3} = 4\sqrt{3}$ and
Applying the Pythagorean Theorem to ΔFAD,

$$AD^2 = 9^2 + \left(4\sqrt{3}\right)^2 = 81 + 48 = 129 \Rightarrow AD = \underline{\sqrt{\mathbf{129}}}$$



C) Since both θ and 2θ are angles in ΔABC, $\theta < 90°$.

$$\sin\theta = \frac{\sqrt{7}}{4} \Rightarrow \cos\theta = +\sqrt{1 - \left(\frac{\sqrt{7}}{4}\right)^2} = \sqrt{\frac{9}{16}} = \frac{3}{4}$$

$$\sin A = \sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{\sqrt{7}}{4}\right)\cdot\frac{3}{4} = \frac{3\sqrt{7}}{8}$$

$$\cos B = \cos^2 C \Rightarrow \cos B = 1 - \sin^2\theta = 1 - \frac{7}{16} = \frac{9}{16}$$

$$\sin B = +\sqrt{1 - \left(\frac{9}{16}\right)^2} = \sqrt{\frac{256 - 81}{16^2}} = \sqrt{\frac{175}{16^2}} = \frac{5}{16}\sqrt{7}$$. Applying the Law of Sines,

$$\frac{\sin A}{BC} = \frac{\sin B}{AC} = \frac{\sin C}{AB} \Rightarrow \frac{3\sqrt{7}}{8BC} = \frac{5\sqrt{7}}{16\cdot 20} = \frac{3}{4AB} \Rightarrow (AB, BC) = (16, 24)$$

Thus, the area of ΔABC is $\frac{1}{2}ac\sin B = \frac{1}{2}\cdot 24\cdot 16\cdot\frac{5\sqrt{7}}{16} = 60\sqrt{7} \Rightarrow (K, L) = \underline{\mathbf{(60, 7)}}$.

Solution #2 (Applying the lesser known formula for the area of a triangle: $\frac{b^2}{2}\cdot\frac{\sin A\sin C}{\sin B}$)

$$\frac{b^2}{2}\cdot\frac{\sin\theta\sin 2\theta}{\sin B} = \frac{b^2\sin^2\theta\cos\theta}{\sin B} = \frac{400\left(\frac{7}{\cancel{16}}\right)\left(\frac{3}{4}\right)}{\frac{5}{\cancel{16}}\sqrt{7}} = \frac{20\cdot 3\cdot 7}{\sqrt{7}} = 60\sqrt{7}.$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2015
ROUND 2 ARITHMETIC/NUMBER THEORY

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) Compute the following sum and express your answer as a base 10 integer.

$$3725_8 + 452_6$$

B) The sum of two primes *A* and *B*, where *A* > *B*, is 60. Compute the average of all possible differences *A* – *B*.

C) Suppose you have a graphing calculator that displays 9 digits.
The largest perfect square that can be displayed is 999, 9_ _ , _ _ _.
Compute the sum of the 5 missing digits, given that $\sqrt{10} \approx 3.16228$, rounded to the nearest 5 decimal places.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2015 SOLUTION KEY

Round 2

A) $3725_8 + 452_6 = 3(8)^3 + 7(8)^2 + 2(8)^1 + 5 = 3(512) + 7(64) + 16 + 5 = 1536 + 448 + 21 = 2005$

$452_6 = 4(6)^2 + 5(6)^1 + 2 = 4(36) + 30 + 2 = 144 + 32 = 176$

Thus, the sum is **<u>2181</u>**.

B) List the primes that are less than 60: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59
The possible ordered pairs for (A, B) are (53, 7), (47, 13), (43, 17), (41, 19), (37, 23) and (31, 29)
The possible differences are 46, 34, 26, 22, 14 and 2.
Therefore the average of the possible differences is 144/6 = **<u>24</u>**.

C) Since 999,999,999 + 1 = 1,000,000,000 = 10^9 and we are searching for the largest integer N for which $N^2 < 10^9$, we note that

$N = \sqrt{10^9} = 10^4 \cdot \sqrt{10} < 10^4(3.1622\cdots) \Rightarrow 31622^2 = 999{,}9\_\ \_{,}\_\ \_\ \_$

$31622^2 = (31600 + 22)^2 = (31600)^2 + 2(22)(31600) + 22^2$

Since we only need the 5 least significant digits, the first term contributes 60000, the second term contributes 90400 and the third term contributes 00484. Summing, the rightmost five digits are 50884 $\Rightarrow$ a digit sum of **<u>25</u>**.

Alternate Solution (submitted by the famous Roman mathematician Brutus Forcus]
Square 31622 by hand (arghhhhh!!!).

$$\begin{array}{r}
31622 \\
\underline{31622} \\
63244 \\
632440 \\
18973200 \\
31622000 \\
\underline{948660000} \\
9999\boldsymbol{\mathit{50884}}
\end{array}$$

Since $\sqrt{10} \approx 3.16228$, the $\sqrt{10}$ is “sandwiched” between 3.1622 and 3.1623, and the next best estimate is 8/10 of the distance between these two values. Since $\sqrt{N}$ increases as N increases, 31623^2 will be a larger value, but will it exceed 1,000,000,000 and overflow the display???

$31623^2 = (31622+1)^2 = \boxed{31622^2} + 2(31622) + 1$, an increase of 63245 over the value calculated above. This increase will push the total over 9 digits.

Thus, 31622^2 is the largest displayable perfect square and the required sum is **<u>25</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2015
ROUND 3 COORDINATE GEOMETRY OF LINES AND CIRCLES

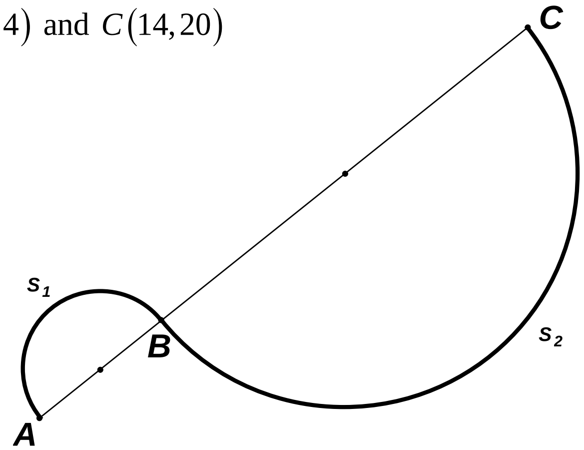
ANSWERS

A) ______________________________

B) (_____________ , _____________)

C) ______________________________

A) S_1 and S_2 are semi-circles. Compute the distance from *A* to *C*, passing through *B* and moving along the circular arcs, given that $A(2,4)$ and $C(14,20)$ and $AB:BC=1:3$.

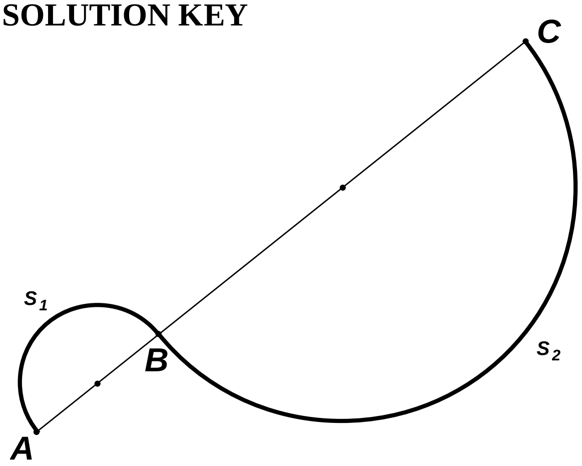


B) The perpendicular bisector of the segment connecting $A(-2,-9)$ and $B(8,-5)$ is $ax+2y=k$. Determine the ordered pair (a,k).

C) The point $C(h,k)$ is the center of the circle $x^2+y^2-10x-4y-140=0$. Point $P(a,b)$, where *a* and *b* are <u>positive</u> integers and $a>b$, is in the <u>exterior</u> of the given circle. If *PC* has a <u>minimum</u> value, compute <u>all</u> possible values of $(h+a)^2+(k+b)^2$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2015 SOLUTION KEY

Round 3



A) Given: $A(2,4), C(14,20)$ and $AB:BC=1:3$.

$AC^2=12^2+16^2=400 \Rightarrow AC=20$

$\Rightarrow (AB,BC)=(5,15)$

$\Rightarrow (r_1,r_2)=\left(\frac{5}{2},\frac{15}{2}\right)$

$\Rightarrow S_1+S_2=\pi\left(\frac{5}{2}+\frac{15}{2}\right)=\underline{\boldsymbol{10\pi}}$

Note: The numerical value of the ratio $AB:BC$ is irrelevant. *B* could be any point between *A* and *C*. If $AB=a$ and $BC=b$, then $m\left(\overset{\frown}{AB}\right)+m\left(\overset{\frown}{BC}\right)=\frac{a\pi}{2}+\frac{b\pi}{2}=\frac{\pi}{2}(a+b)=\frac{\pi}{2}(20)=\underline{\boldsymbol{10\pi}}$. In fact, if $a=0$, then *A* and *B* are the same point, the required distance is a semi-circle on $\overline{AC}$ and again we have $\underline{1\boldsymbol{0\pi}}$.

B) $A(-2,-9)$ and $B(8,-5)$

Since the slope of $\overline{AB}$ is $\frac{-5-(-9)}{8-(-2)}=\frac{4}{10}=\frac{2}{5}$, the slope of the perpendicular bisector is $\frac{-5}{2}$.

Since the slope of $ax+2y=k$ is $\frac{-a}{2}$, we have a = 5.

The midpoint of $\overline{AB}$ is $\left(\frac{-2+8}{2},\frac{-9+(-5)}{2}\right)=(3,-7)$

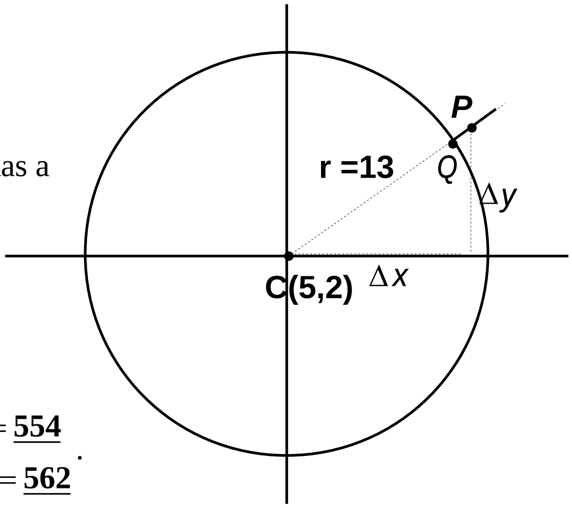
Substituting in $5x+2y=k \Rightarrow k=5(3)+2(-7)=1 \Rightarrow (a,k)=\underline{(\mathbf{5,1})}$.

C) $x^2+y^2-10x-4y-140=0 \Leftrightarrow (x-5)^2+(y-2)^2=169$

$\Rightarrow$ Center: $(h,k)=(5,2)$, Radius: $r=CQ=13$

We require integers Δx and Δy for which $(\Delta x)^2+(\Delta y)^2$ has a value as close as possible to 169.
Examining the integer perfect squares 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, we have $1+169=170$ and $49+121=170$. Clearly, no other smaller integer value is greater than 169. Therefore, $(\Delta x,\Delta y)=(13,1)$, $(11,7)$ and

$(a,b)=(18,3)$ or $(16,9)$ $\Rightarrow (h+a)^2+(k+b)^2=\begin{cases}23^2+5^2=\underline{\mathbf{554}}\\21^2+11^2=\underline{\mathbf{562}}\end{cases}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2015
ROUND 4 ALG 2: LOG & EXPONENTIAL FUNCTIONS

ANSWERS

A) ______________ : ______________

B) (______________ , ______________)

C) (______________ , ______________)

A) Given: $\log_3\left(\log_3 3x\right) = 2 = \log_9(3y)$.
Compute the ratio $y : x$.

B) The graph of $y = \log_8 x$ has an x-intercept at (1, 0), but no y-intercept. We say the graph is asymptotic to the y-axis, that is, the distance between points on the graph and the y-axis get arbitrarily small, but never actually reach zero. Compute the coordinates of point P on the graph of $y = \log_8 x$ which is 0.25 units from the y-axis.

C) Given $\log_{14}\left(0.125\right) = W$, $\log_8\left(49\right)$ may be expressed in terms of W as $m\left(\frac{W+b}{W}\right)$, for <u>constants</u> m and b, where $b > 0$. Compute the ordered pair $\left(m, b\right)$.

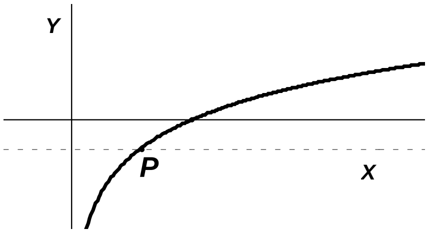
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2015 SOLUTION KEY

Round 4

A) $\log_3(3x)=9 \Rightarrow 3x=3^9 \Rightarrow x=3^8,\ 3y=81 \Rightarrow y=27$ Thus, $\dfrac{y}{x}=\dfrac{3^3}{3^8}=\dfrac{1}{3^5}=\dfrac{1}{243}\Rightarrow$ **<u>1 : 243</u>**.

B) Consider the horizontal line $y = k$. For any value of k, it intersects $y=\log_8 x$ exactly once at point P. We require the distance from P to the y-axis to be 0.25, but this is simply the x-coordinate of the point P. Thus, we have

$$k=\log_8\left(0.25\right)\Leftrightarrow 8^k=2^{-2}\Rightarrow k=-\frac{2}{3}\Rightarrow \underline{P\left(\mathbf{\frac{1}{4},-\frac{2}{3}}\right)}.$$



C) Suppose $\log_8 49 = N$. Then: $8^N=49\Leftrightarrow 2^{3N}=7^2$ Taking the log of both sides,

$$3N\log 2=2\log 7\Rightarrow N=\frac{2}{3}\cdot\frac{\log 7}{\log 2}\Rightarrow \boxed{\log_8 49=\frac{2}{3}\log_2 7}.$$

$$W=\log_{14}0.125=\log_{14}\frac{1}{8}=\log_{14}2^{-3}=-3\log_{14}2=-3\cdot\frac{\log 2}{\log 14}=\frac{-3\log 2}{\log 2+\log 7}=\frac{-3}{1+\dfrac{\log 7}{\log 2}}=\frac{-3}{1+\log_2 7}$$

Cross multiplying, $W+W\log_2 7=-3\Rightarrow \log_2 7=\dfrac{-3-W}{W}$

Substituting, $\log_8 49=\dfrac{2}{3}\left(\dfrac{-3-W}{W}\right)=-\dfrac{2}{3}\left(\dfrac{W+3}{W}\right)\Rightarrow (m,b)=\underline{\left(\mathbf{-\frac{2}{3},3}\right)}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2015
ROUND 5 ALG 1: RATIO, PROPORTION OR VARIATION

ANSWERS

A) ________________________________

B) (______________ , ______________)

C) (______________ , ______________)

A) Given: $x+2=y+1=a$ and $\frac{x}{y}=\frac{a}{b}$.
If $x+y=6$, compute the numerical value of *b*.

B) $\frac{2n^2+13n-24}{2n^3-8n}$ is a <u>nonzero</u> <u>defined</u> ratio.
There are *K* values that cause the ratio to be zero or undefined and the smallest value is *J*.
Compute the ordered pair (*K*, *J*).

C) For positive integers *a*, *b* and *k*, $\frac{3a+7}{b+2}=\frac{5}{6}$, when *b* = *ka* and *a* > 1.
Compute the ordered pair (*a*, *b*).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2015 SOLUTION KEY

Round 5

A) Given: $x+2=y+1=a$, $\frac{x}{y}=\frac{a}{b}$ and $x+y=6$.

$$x+y=6 \Leftrightarrow (a-2)+(a-1)=6 \Rightarrow a=4.5,\ x=2.5,\ y=3.5$$

$$\frac{2.5}{3.5}=\frac{4.5}{b} \Leftrightarrow \frac{5}{7}=\frac{9}{2b} \Rightarrow 10b=63 \Rightarrow b=\underline{\mathbf{\frac{63}{10}}} \text{ or } \underline{\mathbf{6.3}}.$$

B) $\frac{2n^2+13n-24}{2n^3-8n}=\frac{(2n-3)(n+8)}{2n(n+2)(n-2)}$.

The ratio is zero when the numerator is zero, namely when $n=\frac{3}{2}$ or -8 .

The ratio is undefined when the denominator is zero, namely when $n=0$ or ± 2.

Therefore, $(K,J)=\underline{\mathbf{(5,-8)}}$.

C) Given: $\frac{3a+7}{b+2}=\frac{5}{6}$ and $b=ka$ Substituting for *b* and cross multiplying, we have

$18a+42=5ka+10 \Rightarrow a=\frac{32}{5k-18}$ and $5k-18$ must be a factor of 32. Thus,

$5k-18=1,2,4,8,16$ or $32 \Rightarrow 5k=19,20,22,26,34$ or 50.

The only possible integer values of *k* are 4 or $\cancel{10}$ (10 is rejected since $k=10 \Rightarrow a=1$.)

$k=4 \Rightarrow a=32/2=16$ and $b=4a \Rightarrow$ $(a,b)=\underline{\mathbf{(16,64)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2015
ROUND 6 PLANE GEOMETRY: POLYGONS (no areas)

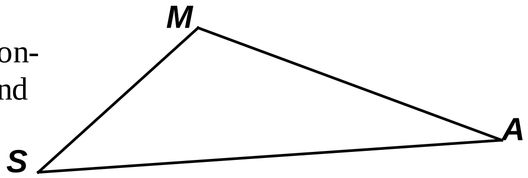
ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) The triangle at the right would most likely be called ΔSAM by someone whose first language is English. However, a non-English speaking student could have started at any vertex and listed the vertices clockwise or counterclockwise, giving many more possible names.
How many different names are possible for polygon DUMBWAITER that begin with a vowel?



B) A concave hexagon F has 2 angles measuring 150° and 165°. The remaining 4 angles have measures in a ratio of $1 : 2 : 4 : 8$. Compute the measure of the <u>largest</u> interior angle in F.
Note: Since the hexagon is concave, one of the interior angles is reflexive, i.e. its measure is between 180° and 360°.

C) If a regular polygon had 6 more sides, its exterior angles would each be decreased by 3°. Compute the measure of an interior angle of the original regular polygon.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2015 SOLUTION KEY

Round 6

A) The name of the polygon must begin with A, E, I or U, i.e. 4 choices and proceed through the letters successively either clockwise or counterclockwise. Thus, $4\cdot 2=\underline{\mathbf{8}}$.

B) $150+165+x+2x+4x+8x=180\left(6-2\right)=720$
$\Rightarrow 15x=720-315=405\Rightarrow x=27$.
The largest interior angle is $27\cdot 8=\underline{\mathbf{216}}$.

C) $\dfrac{360}{n}-\dfrac{360}{n+6}=3\Leftrightarrow\dfrac{120}{n}-\dfrac{120}{n+6}=1\Leftrightarrow 120n+720-120n=n(n+6)$
$\Leftrightarrow n(n+6)=720=72(10)=24(30)$.
As the last factorization shows, $n = 24$ is the positive solution to the quadratic equation.
In a regular 24-gon, the exterior angle measures 15°; therefore, the interior angle is $\underline{\mathbf{165}}$°.

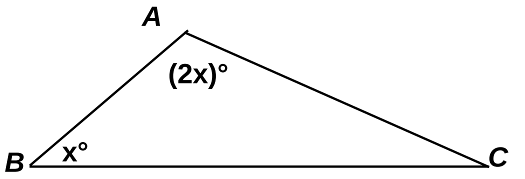
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2015
ROUND 7 TEAM QUESTIONS
ANSWERS

A) ____________________________ D) ____________________________

B) ____________________________ E) (__________ , __________)

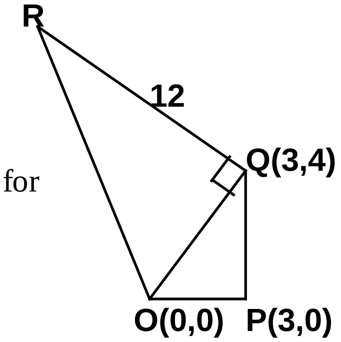
C) (__________ , __________) F) ____________________________

A) In ΔABC, with angles as indicated, D lies on $\overline{BC}$ such that $\overrightarrow{AD}$ bisects $\angle A$. $\frac{DC}{DA} = 2$ and $BC = 6$. Compute the perimeter of ΔABC.



B) Consider the following statements about the integer x.
x and $x + 2$ are prime.
$x - 1$ is a multiple of 5. $x + 3$ is a multiple of 7.
The minimum positive integer value of x for which all of these statements are true is $x = 11$. Determine the next largest value of x for which all of these statements are true.

C) Compute the coordinates of point R.



D) Given: $a, b, m, n > 0$, $a^{b+c} = m$ and $b^{c+a} = n$.
If $\log\left(a^b b^a\right) = 2$, express $(ab)^c$ as a simplified expression in terms of m and n.

E) List the integers from 1 to 100 inclusive in 10 rows as indicated at the right.
Let n be the smallest prime in the list.
Repeat the following pair of statements until $n^2 > 100$.
Cross out every n^{th} number in the list which is larger than n.
Now let n be the smallest integer in the list not crossed out.

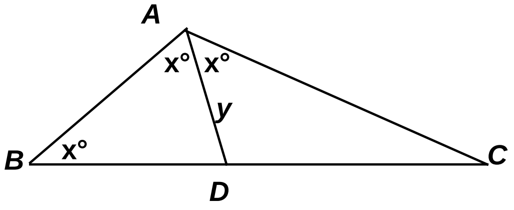
1	2	3	…	10
11	12	13	…	20
21	22	23		30
…				
81	82	83	…	90
91	92	93	…	100

Let (a, b) be consecutive un-crossed-out integers, where $b > a$.
The simplified ratio of the number of ordered pairs for which $b - a = 2$ to the number of ordered pairs for which $b - a = 6$ is $k : j$.
Determine the ordered pair of integers (k, j).

F) A polygon with n sides has more than 1,000,000 diagonals. What is the minimum number of sides this polygon could have?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2015 SOLUTION KEY

Team Round



A) ΔABD is isosceles, so let $DB = DA = y$ and $DC = 2y$.
Using the angle bisector theorem on ΔABC,
$\frac{AC}{AB} = \frac{DC}{DB} = \frac{2y}{y} = 2$. $BC = 6 \Rightarrow y = 2$.
Let $AB = k$ and $AC = 2k$. Using the Law of Cosines,
on $\angle ABD$ in ΔABD: $2^2 = k^2 + 2^2 - 4k\cos\left(x^\circ\right) \Rightarrow 4k\cos\left(x^\circ\right) = k^2 \Rightarrow \cos\left(x^\circ\right) = \frac{k}{4}$ $(k \neq 0)$
on $\angle DAC$ in ΔDAC: $4^2 = 2^2 + 4k^2 - 8k\cos\left(x^\circ\right)$. Substituting for $\cos\left(x^\circ\right)$,
$12 = 4k^2 - 8k\left(\frac{k}{4}\right) = 2k^2 \Rightarrow k = \sqrt{6}$. The perimeter of ΔABC is $3(y + k) \Rightarrow$ $\underline{\mathbf{3\left(2+\sqrt{6}\right)}}$.

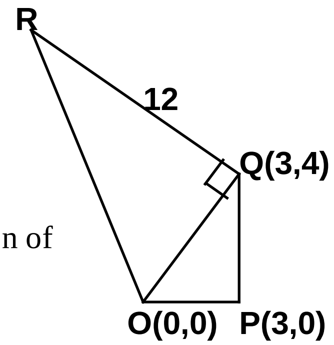
B) Let $x = 5k + 1$ and $x + 2 = 7j - 1$ so that $x - 1$ and $x + 3$ will be multiples of 5 and 7 respectively. Then: $(x+3) - (x-1) = 4 = 7j - 5k$
Construct a table of (k, j) values satisfying this relation.
Since the linear relation $4 = 7j - 5k$ or $k = \frac{7j-4}{5}$ has a <u>slope</u> of 7/5, once we find an initial pair of (k, j) values, subsequent pairs are easily determined. $(k, j) = (2, 2)$ is our initial pair.

	X	k	j	x + 2
1	11	2	2	13
2	46	9	7	48
3	81	16	12	83
4	116	23	17	118

Since both x and $x + 2$ are even in even rows (and therefore not prime), we consider only odd rows. The x-values in odd rows are of the form $70n + 11$, where $n = 0, 1, 2, \ldots$.
$n = 0$ gives us the first pair of primes $(x, x+2) = (11, 13)$
We try successive values of n until both x and $x + 2$ are again prime.
$n = 1 \Rightarrow$ (81, 83) rejected (81 is not prime)
$n = 2 \Rightarrow$ (151, 153) rejected (153 is divisible by 3)
$n = 3 \Rightarrow$ (221, 223) rejected ($221 = 13\cdot17$)
$n = 4 \Rightarrow$ (291, 293) rejected (291 is divisible by 3)
$n = 5 \Rightarrow$ (361, 363) rejected ($361 = 19^2$)
$n = 6 \Rightarrow$ (**<u>431</u>**, 433) Bingo! – both are prime
Both numbers must be checked for divisibility by primes smaller than their square root.
Since $21^2 = 441$ is larger than both numbers, we need check for divisibility by only 8 primes - 2, 3, 5, 7, 11, 13, 17 and 19.
There are well-known rules for 2, 3, 5 and 11.
Brute force suffices for 7, 13, 17 and 19.
The details of the divisibility check are left to you.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2015 SOLUTION KEY

Team Round - continued



C) $OQ = 5 \Rightarrow OR = 13$

An equation of a origin-centered circle passing through point R is $x^2 + y^2 = 169$

Since the slope of $\overline{OQ}$ is $\frac{4}{3}$, the slope of $\overleftrightarrow{RQ}$ is $-\frac{3}{4}$ and the equation of $\overleftrightarrow{RQ}$ is $(y-4) = -\frac{3}{4}(x-3) \Leftrightarrow 3x + 4y = 25$

Solving these equations, $\begin{array}{l} 9x^2 + 9y^2 = 9 \cdot 169 \\ 9x^2 = (25-4y)^2 \end{array} \Rightarrow \left(625 - 200y + 16y^2\right) + 9y^2 = 9 \cdot 169$

$\Rightarrow\ 25y^2 - 200y + 625 - 1521 = 0$

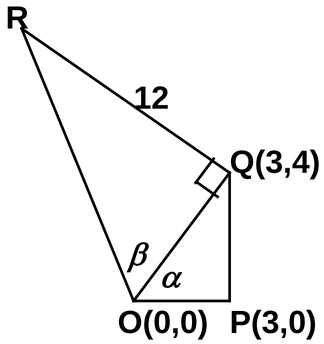
$\Rightarrow\ 25y^2 - 200y - 896 = 0$

$\Rightarrow\ y = \dfrac{200 \pm \sqrt{200^2 + 4 \cdot 25 \cdot (896)}}{50} = \dfrac{200 \pm \sqrt{100(400+896)}}{50}$

$\Rightarrow\ y = \dfrac{200 \pm \sqrt{10^2 \cdot 36^2}}{50} = \dfrac{200 \pm 360}{50} \Rightarrow y = \dfrac{56}{5} = 11.2,\ \cancel{-\dfrac{16}{5}}$

Substituting, $3x + 4(11.2) = 25 \Rightarrow 3x = -19.8 \Rightarrow x = -6.6$

Thus, $R\underline{\left(\mathbf{-6.6,\ 11.2}\right)}$ or $R\underline{\left(-\dfrac{\mathbf{33}}{\mathbf{5}}, \dfrac{\mathbf{56}}{\mathbf{5}}\right)}$.



Solution #2 (Norm Swanson):

Clearly, in ΔPOQ $\sin\alpha = \frac{4}{5}$ and $\cos\alpha = \frac{3}{5}$, and,

in ΔROQ, $\sin\beta = \frac{12}{13}$ and $\cos\beta = \frac{5}{13}$

Since the coordinates of point R are $(\cos(\angle ROP), \sin(\angle ROP))$, we must evaluate $\sin(\alpha + B)$ and $\cos(\alpha + B)$.

$\sin(\alpha+\beta) = \frac{4}{5} \cdot \frac{5}{13} + \frac{3}{5} \cdot \frac{12}{13} = \frac{56}{65}$ $\quad \cos(\alpha+\beta) = \frac{3}{5} \cdot \frac{5}{13} - \frac{4}{5} \cdot \frac{12}{13} = -\frac{33}{65} \Rightarrow R\underline{\left(-\dfrac{\mathbf{33}}{\mathbf{65}}, \dfrac{\mathbf{56}}{\mathbf{65}}\right)}$

D) Taking log of both sides, $\begin{cases} \log m = (b+c)\log a \\ \log n = (c+a)\log b \end{cases}$

Adding, we have

$\log m + \log n = \log mn = c(\log a + \log b) + a\log b + b\log a = \log(ab)^c + \log\left(a^b b^a\right) = \log(ab)^c + 2$

$\Rightarrow\ \log mn = \log(ab)^c + \log 100 = \log\left(100(ab)^c\right) \Rightarrow\ (ab)^c = \underline{\dfrac{\mathbf{mn}}{\mathbf{100}}}$

Team Round - continued

E) All multiples of 2, 3, 5 and 7 are crossed out. Since every composite number less than or equal to 100 is divisible by at least one of these numbers, only the primes remain.

$b-a=2 \Rightarrow (3,5), (5,7), (11,13), (17,19), (29,31), (41,43), (59,61), (71,73)$ - 8 Pairs

$b-a=6 \Rightarrow (23,29), (31,37), (47,53), (53,59), (61,67), (73,79), (83,89)$ - 7 pairs

Therefore, $(k, j) = \underline{(\mathbf{8,7})}$.

F) $\dfrac{n(n-3)}{2} > 1,000,000 \Rightarrow n(n-3) > 2(10^6)$.

Since $n > n-3$, $n^2 > n(n-3)$. Therefore, $n > \sqrt{2(10^6)} = 10^3\sqrt{2}$.

If we know that $\sqrt{2} \approx 1.414$, our job is a lot easier.
We start with $n = 1415$.
$1415(1412) = 1,997,980$ and this product is just a little too small.
$1416(1413) = 2,000,808$ and we have the minimum, n = **<u>1416</u>.**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2015 ANSWERS

Round 1 Trig: Right Triangles, Laws of Sine and Cosine

A) (188, 235) B) $\sqrt{129}$ C) $\left(60, 7\right)$

Round 2 Arithmetic/Elementary Number Theory

A) 2181 B) 24 C) 25 [50884]

Round 3 Coordinate Geometry of Lines and Circles

A) 10π B) $\left(5, 1\right)$ C) 554, 562

Round 4 Algebra 2: Log and Exponential Functions

A) 1 : 243 B) $\left(\frac{1}{4}, -\frac{2}{3}\right)$ C) $\left(-\frac{2}{3}, 3\right)$

Round 5 Algebra 1: Ratio, Proportion or Variation

A) 6.3 (or $\frac{63}{10}$) B) (5, –8) C) (16, 64)

Round 6 Plane Geometry: Polygons (no areas)

A) 8 B) 216 C) 165

Team Round

A) $3\left(2+\sqrt{6}\right)$

B) 431

C) $\left(-6.6, 11.2\right)$ or $\left(-\frac{33}{5}, \frac{56}{5}\right)$

D) $\frac{mn}{100}$

E) $\left(8, 7\right)$

F) 1416