MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016
ROUND 1 VOLUME & SURFACES

ANSWERS

A) ______________________________

B) ______________________________

C) (_____________ , _____________)

A) The sum of the lengths of the edges of a cube is 60 inches.
Compute the surface area of this cube (in inches^2).

B) The dimensions of a rectangular solid are x, y, and xy.
If the interior diagonal of this solid has length $xy + 1$, find *all* possible expressions for y in terms of x.

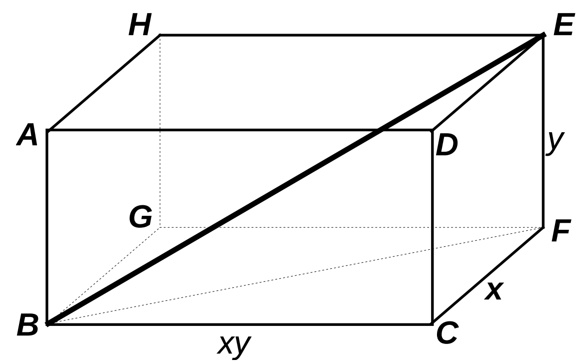
C) Three faces of a rectangular solid have areas of 864, 1440 and 2160 units^2, respectively.
This solid is packed with k congruent cubes with edge E, where k is as <u>small</u> as possible.
Each of the k cubes is inscribed in a sphere. The total volume of the E spheres, in simplified form, is $A\pi\sqrt{B}$. Compute the ordered pair (A, B).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016 SOLUTION KEY

Round 1

A) Since a cube has 12 edges, each edge has length 5.
Thus, each face of the cube is a square of side 5 and the surface area is $6\left(5^2\right) = \underline{\mathbf{150}}$.

B) In ΔBFC, $BF^2 = (xy)^2 + x^2$.
In ΔBEF, $BE^2 = BF^2 + FE^2 = x^2y^2 + x^2 + y^2$.
Equating, $x^2y^2 + x^2 + y^2 = (xy+1)^2$
$\Rightarrow \cancel{x^2y^2} + x^2 + y^2 = \cancel{x^2y^2} + 2xy + 1$
$\Rightarrow x^2 - 2xy + y^2 = (x-y)^2 = 1$
$\Rightarrow x - y = \pm 1 \Rightarrow y = \underline{\mathbf{x+1}}$ or $y = \underline{\mathbf{x-1}}$
Relationship discovered by Grant Landon (Miles River Middle School in Hamilton, MA)



FYI: Generalization - If the dimensions of a rectangular solid are x, y and $\frac{xy}{k}$ and the interior diagonal has length $\frac{xy}{k} + k$ then $y = x + k$ or $y = x - k$.
Proof of the generalization is left to you. Here is a numerical check.
1) Suppose a solid has dimensions 9, 12 and 36. How long is the interior diagonal?
$\frac{9 \cdot 12}{3} = 36$ and $12 = 9 + 3 \Rightarrow$ interior diagonal $= 36 + 3 = 39$.
2) If a solid has dimensions 5, 8, and $\frac{40}{3}$, then $(x, y, k) = (5, 8, 3)$ and the interior diagonal has length $\frac{40}{3} + 3 = \frac{49}{3}$.

C) Assume the dimensions of the solid are L, W and H. Then: $\begin{cases} (1)\ \ LW = 864 \\ (2)\ \ LH = 1440 \\ (3)\ \ WH = 2160 \end{cases}$

Dividing (3) and (2), $\frac{W}{L} = \frac{216}{144} = \frac{72(3)}{72(2)} \Rightarrow W = \frac{3}{2}L$.

Substituting in (1), $\frac{3}{2}L^2 = 864 \Rightarrow \frac{L^2}{2} = 288 \Rightarrow L^2 = 2^2(144) \Rightarrow L = 24 \Rightarrow W = 36$, $H = 60$

Since the GCF(24,36,60) is 12, the largest cube that can be packed inside the original solid has edge 12. Thus, we will need $2 \cdot 3 \cdot 5 = 30$ of these cubes and $(k, E) = (30, 12)$.

The diameter of each sphere will be the diagonal of the cube (which has length $12\sqrt{3}$).
Thus, the total volume of the 30 spheres is
$30 \cdot \frac{4}{3}\pi\left(6\sqrt{3}\right)^3 = 40\pi \cdot 216\left(3\sqrt{3}\right) = 25920\pi\sqrt{3} \Rightarrow (A, B) = \underline{\mathbf{(25920, 3)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016
ROUND 2 PYTHAGOREAN RELATIONS IN RECTILINEAR FIGURES

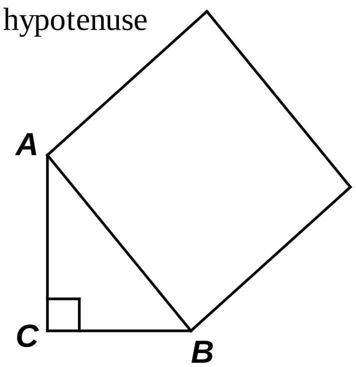
ANSWERS

A) ________________________________

B) ________________________________

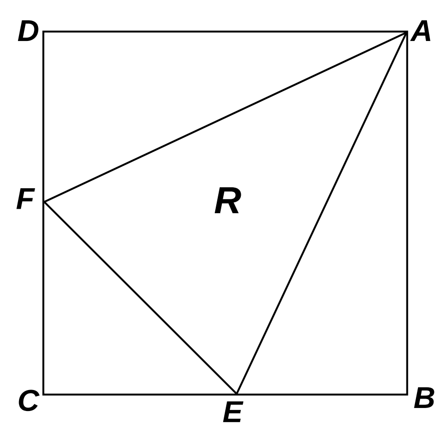
C) ________________________________

A) The legs of right triangle *ABC* have lengths 7 and x. A square drawn on the hypotenuse has area $x + 69$.
Compute the area of the square drawn on the shorter leg.



B) In right triangle *ABC*, the short leg has length 48 and the difference between the lengths of the long leg and the hypotenuse is 18. Compute the perimeter of this triangle.

C) In square *ABCD*, *E* and *F* are the midpoints of $\overline{BC}$ and $\overline{CD}$, respectively, and the area of ΔAEF is *R* square units. Find the length of the altitude of ΔAEF from *A* as a <u>simplified</u> expression in terms of *R*.

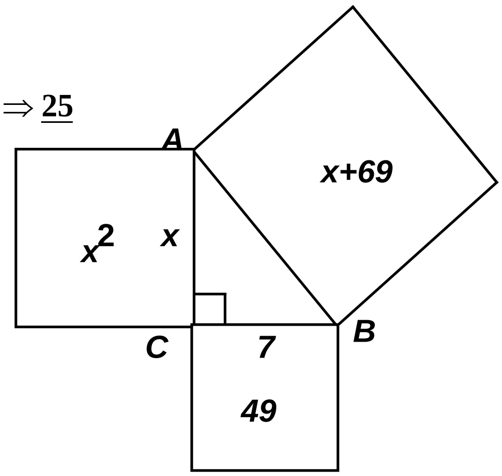


MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016 SOLUTION KEY

Round 2

A) $x^2+49=x+69 \Leftrightarrow x^2-x-20=(x-5)(x+4)=0 \Rightarrow x=5 \Rightarrow \underline{\mathbf{25}}$

.



B) $48^2+\cancel{x^2}=(x+18)^2=\cancel{x^2}+36x+18^2 \Rightarrow x=\dfrac{48^2-18^2}{36}=\dfrac{(48+18)(48-18)}{36}=\dfrac{66\cdot 30}{36}=11\cdot 5=55$.

Thus, the perimeter is $48+55+73=\underline{\mathbf{176}}$.

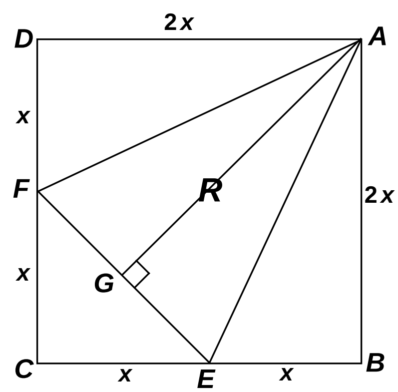
C) Let AG denote the required altitude and the length of the side of the square be $2x$.
Then $R = 4x^2 -$ area($\Delta ADF + \Delta FCE + \Delta ABE$)

$= 4x^2 - x^2 - x^2/2 - x^2 = 3x^2/2 \Rightarrow x^2=\dfrac{2R}{3} \Rightarrow x=\sqrt{\dfrac{2R}{3}}$

Since $EF = x\sqrt{2}$, we have $\dfrac{1}{2}\cdot x\sqrt{2}\cdot AG=R$.

Substituting for x, $\dfrac{1}{2}\cdot\sqrt{\dfrac{2R}{3}}\cdot\sqrt{2}\cdot AG=R \Rightarrow \sqrt{\dfrac{R}{3}}\cdot AG=R$

$\Rightarrow AG = R\cdot\sqrt{\dfrac{3}{R}}=\sqrt{\dfrac{3R^2}{R}}=\underline{\sqrt{\mathbf{3R}}}$ (since $R\neq 0$).



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016
ROUND 3 ALG 1: LINEAR EQUATIONS

ANSWERS

A) ______________________________

B) __________________________ mph

C) (_____________ , _____________)

A) The sum of five <u>consecutive</u> integers n_1, n_2, n_3, n_4, n_5, where $n_1 < n_2 < n_3 < n_4 < n_5$ is 95. Compute n_4.

B) At 50 mph it takes 15 minutes less time to reach my destination than it would if I travelled only 40 mph. How fast must I travel (in miles per hour) to reach the same destination in 40 minutes?

C) The Chicago White Sox baseball team hired a new coach to help improve their dreadful base stealing statistics. To date, as a team, they have succeeded in stealing 27 times and failed x times. The management's minimum acceptable success rate is 80%. Suppose the team attempts 50 more steals by the All-star break and they get caught stealing k times. Determine the ordered pair (x, k), where x has a maximum value, but the team's base stealing success rate is <u>exactly</u> 80%.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016 SOLUTION KEY

Round 3

A) Solution #1: (Brute Force)

$n_1+n_2+n_3+n_4+n_5 = x+(x+1)+(x+2)+\boxed{(x+3)}+(x+4)=5x+10=95 \Rightarrow x=17$

$\Rightarrow n_4 = x+3 = \underline{\mathbf{20}}$.

Solution #2: (a Little Finesse)

The middle integer $n_3=(x+2)$ is the average of the 5 integers, namely $\frac{95}{5}=19$.

Therefore, $n_4 = n_3+1 = x+3 = \underline{\mathbf{20}}$.

B) 15 minutes = $\frac{1}{4}$ hour $\Rightarrow 40t = 50\left(t-\frac{1}{4}\right) \Leftrightarrow 160t = 200t-50 \Rightarrow t=\frac{5}{4}$

Thus, my destination is 50 miles away.

40 minutes = $\frac{2}{3}$ hour $\Rightarrow r\left(\frac{2}{3}\right)=50 \Leftrightarrow r = \underline{\mathbf{75}}$ mph.

C) $\frac{27+(50-k)}{(27+x)+50}=\frac{4}{5} \Leftrightarrow \frac{77-k}{77+x}=\frac{4}{5} \Rightarrow 4x+5k=77$

$\Rightarrow\ x=\frac{77-5k}{4}=\frac{76-4k}{4}+\frac{1-k}{4}=19-k+\frac{1-k}{4}$

Clearly, to insure that *x* and *k* are integers, *k* must increase by 4.
However, this decreases the value of *x*.
Thus, $(x,k)=\underline{(\mathbf{18,1})}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016
ROUND 4 ALG 1: FRACTIONS & MIXED NUMBERS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) A man takes a bus trip covering 25 miles in 30 minutes. He then travels 3020 miles by plane in 3 hours. Compute his average rate (in miles per hour) for the entire trip. If necessary, round your answer to the nearest integer.

B) A carpenter and his assistant are building a large shed. They work together for 7 days and then the carpenter leaves the two-man team and the assistant finishes the job. If 70% of the job was completed before the carpenter left and the carpenter works twice as fast as the assistant, how many more days will it take the assistant to finish the job?

C) Compute the value of $x = 1 + \cfrac{2}{3 + \cfrac{4}{1 + \cfrac{2}{3 + \cfrac{4}{1 + \ldots}}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016 SOLUTION KEY

Round 4

A) $R = \dfrac{D_1 + D_2}{T_1 + T_2} = \dfrac{25 + 3020}{3 + .5} = \dfrac{3045}{7/2} = \dfrac{6090}{7} = \underline{\mathbf{870}}$ mph.

B) Assume carpenter can complete the job alone in x days, while the assistant would take $2x$ days. Then:

Their rates are $\dfrac{1}{x}$ and $\dfrac{1}{2x}$ respectively, implying $7\left(\dfrac{1}{x} + \dfrac{1}{2x}\right) = \dfrac{7}{10} \Rightarrow \dfrac{3}{2x} = \dfrac{1}{10} \Rightarrow x = 15$.

If it takes the assistant T days to complete the job, then $\dfrac{1}{30}T = \dfrac{3}{10} \Rightarrow T = \underline{\mathbf{9}}$ days.

C) $x = 1 + \dfrac{2}{3 + \dfrac{4}{x}} \Rightarrow 1 + \dfrac{2}{\dfrac{3x+4}{x}} = 1 + \dfrac{2x}{3x+4} = \dfrac{5x+4}{3x+4}$

$\Rightarrow 3x^2 + 4x = 5x + 4 \Rightarrow 3x^2 - x - 4 = 0 \Leftrightarrow (3x-4)(x+1) = 0 \Rightarrow x = \cancel{-1},\ \underline{\mathbf{\dfrac{4}{3}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016
ROUND 5 INEQUALITIES & ABSOLUTE VALUE

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) For what integer value of *k*, is the inequality $7 \le |x| < k$ satisfied by <u>exactly</u> 6 integers?

B) Compute <u>all</u> values of *x* for which $|3x+1| \le 7-x$.

C) Solve for *x*: $-x^5 + 5x^3 - 4x > 0$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016 SOLUTION KEY

Round 5

A) Graphically, the solution set is

$k = \underline{\mathbf{10}} \Rightarrow -7, -8, -9, 7, 8, 9$

B) Clearly, $x \le 7$, otherwise the right side of the equation is negative and the result would be extraneous.

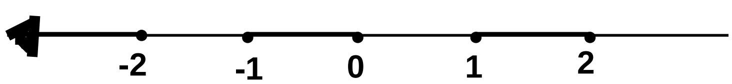
$|3x+1| = \begin{cases} 3x+1 \\ -(3x+1) \end{cases}$ depending on whether $x \ge -\frac{1}{3}$ or $x < -\frac{1}{3}$.

For $x \ge -\frac{1}{3}$, $3x+1 \le 7-x \Rightarrow x \le \frac{3}{2} \Rightarrow -\frac{1}{3} \le x \le \frac{3}{2}$

For $x < -\frac{1}{3}$, $-3x-1 \le 7-x \Rightarrow x \ge -4 \Rightarrow -4 \le x < -\frac{1}{3}$

$\Rightarrow \underline{\mathbf{-4 \le x \le \frac{3}{2}}}$.

C) $-x^5 + 5x^3 - 4x = -x(x^4 - 5x^2 + 4) = -x(x^2 - 1)(x^2 - 4) > 0$
Multiplying by -1 and factoring, $x(x + 1)(x - 1)(x + 2)(x - 2) < 0$, we have critical points at $x = -2, -1, 0, 1$ and 2. At the extreme left, all 5 factors are negative. As we move from left to right, every time a critical point is passed one more factor becomes positive.



So the sign of the product alternates negative and positive. In the highlighted regions, the product is negative, i.e. $\underline{\mathbf{x < -2 \text{ or } -1 < x < 0 \text{ or } 1 < x < 2}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016
ROUND 6 ALG 1: EVALUATIONS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) Compute the value of x. $2+3\cdot 4-x\div 6=7$

B) The rows and columns of the cells on an 8 x 8 grid are each numbered from 1 to 8. A cell is colored if the <u>sum</u> of its row number and column number is divisible by 3 or by 5 (or both). How many such cells are colored?

C) Two sides of a triangle have lengths of 17 and 38. The perimeter is an integer multiple of 4. How many lengths are possible for the third side?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016 SOLUTION KEY

Round 6

A) Invoking the PEMDAS rule, $2+3\cdot 4-x\div 6=7$ is equivalent to $2+(3\cdot 4)-\dfrac{x}{6}=7$

$\Rightarrow 14-\dfrac{x}{6}=7\Rightarrow \dfrac{x}{6}=7\Rightarrow x=\underline{\mathbf{42}}$

B) Sums triggering the paintbrush are 3, 5, 6, 9, 10, 12 and 15, since the smallest sum is 2 and the largest 16. $(r=2)+(c=1...8)\Rightarrow 3,5,6,9,10$ $(r=4)+(c=1...8)\Rightarrow 5,6,9,10,12$
$(r=6)+(c=1...8)\Rightarrow 9,10,12$

All other rows generate 4 triggering sums.
Thus, there will be $5\cdot 4+2\cdot 5+3=\underline{\mathbf{33}}$.

	1	2	3	4	5	6	7	8	
1		x		x	x			x	4
2	x		x	x			x	x	5
3		x	x			x	x		4
4	x	x			x	x		x	5
5	x			x	x		x		4
6			x	x		x			3
7		x	x		x			x	4
8	x	x		x			x		4

C) Let x denote the length of the third side.
According to the triangle inequality, $\begin{cases} x+17>38 \\ 17+38>x \end{cases}\Rightarrow 22\le x\le 54$
The perimeter is 55 + x. The minimum value of x producing a multiple of 4 is 25
The values of x are of the form 25 + 4k. k = 7 produces the largest possible value of x, namely 53.
Thus, there are $\underline{\mathbf{8}}$ possible lengths for the third side.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016
ROUND 7 TEAM QUESTIONS

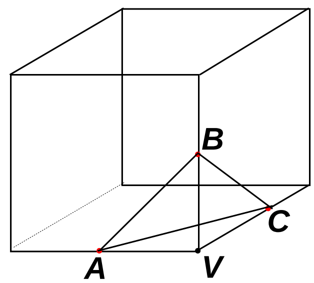
ANSWERS

A) ______________________ D) ______________________

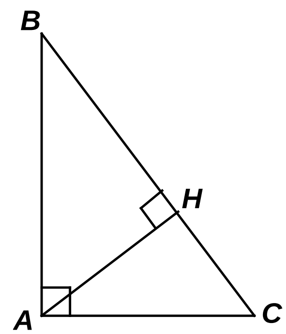
B) ______________________ E) $\{x \mid$ ______________________ $\}$

C) (___ , ___ , ___ , ___ , ___) F) (__________ , __________)

A) Points A, B, and C lie on the edges of a cube.
The distance from vertex V to the plane containing points A, B, and C is $\frac{m}{n}\sqrt{r}$, in simplified radical form.
If $AB = 13,\ BC = 14,$ and $CA = 15$, compute $m+n+r$.



B) Given: ΔABC, with a right angle at A. $AB = 9$, $AC = 2\sqrt{10}$
Let H be the endpoint of the perpendicular segment drawn from A to $\overline{BC}$.
Compute $HB - HC$.



C) Given 5 numbers v, w, x, y, and z.
If the average of three of them is added to the remaining two, the results are: 21, 23, 25, 27, 25, 27, 29, 29, 31, and 33.
Specify the 5 numbers as $(v_1, v_2, v_3, v_4, v_5)$, where $v_1 \le v_2 \le v_3 \le v_4 \le v_5$.

D) Given: $\dfrac{2x+3}{(x-2)^3} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{C}{(x-2)^3}$ Compute $A^2 + B^2 + C^2$.

E) Specify the condition which describes those values of x (and only those values) which satisfy the following inequality: $2\left|x^2-1\right| - \left|x^2+1\right| \le 9x+7$
If necessary, the proper use of the connectors "and" / "or" is required.

F) The sequence of Fibonacci numbers $Fib(n)$ is defined by

$$\begin{cases} Fib(n) = Fib(n-1) + Fib(n-2) \text{ for } n \ge 3 \\ Fib(2) = 2 \\ Fib(1) = 1 \end{cases}$$

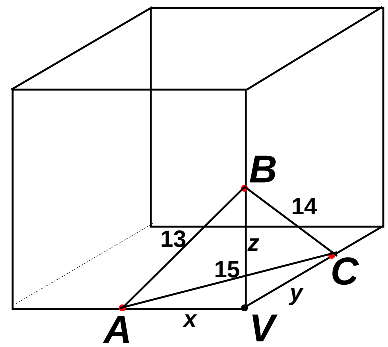
The first five Fibonacci numbers are 1, 2, 3, 5, and 8. An <u>open</u> interval (a,b) is defined to include all values of x satisfying the inequality $a < x < b$.
For some <u>minimum</u> value of $k > 0$, the <u>open</u> interval $(Fib(k), Fib(k+1))$ contains 2 distinct integer perfect cubes j^3 and $(j+1)^3$. Compute the ordered pair (j,k).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016 SOLUTION KEY

Team Round

A) The area of ΔABC may be determined by $\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$, where *a*, *b*, and *c* denote side-lengths and *s* denotes the semi-perimeter (Heron's Formula).

$s = \frac{13+14+15}{2} = 21 \Rightarrow \text{Area} = \sqrt{21\cdot 8\cdot 7\cdot 6} = \sqrt{2^4 3^2 7^2} = 84$



Let *h* denote the distance from *V* to the plane *ABC*. Then:

$\text{Vol} = \frac{1}{3}h(84) = 28h$.

But, using ΔBAV as the base and $\overline{VC}$ as the height,

$\text{Vol} = \frac{1}{3}\left(\frac{1}{2}xy\right)z = \frac{xyz}{6}$

$$\begin{cases} x^2+y^2=225 \\ x^2+z^2=169 \\ y^2+z^2=196 \end{cases} \Rightarrow (x,y,z) = \left(3\sqrt{11}, 3\sqrt{14}, \sqrt{70}\right)$$

Thus, $28h = \frac{3\sqrt{11}\cdot 3\sqrt{14}\cdot\sqrt{70}}{6} = \frac{\cancel{2\cdot 3}\cdot 3\cdot 7\sqrt{5\cdot 11}}{\cancel{6}} = 21\sqrt{55} \Rightarrow h = \frac{3}{4}\sqrt{55} \Rightarrow m+n+r = \underline{\mathbf{62}}$

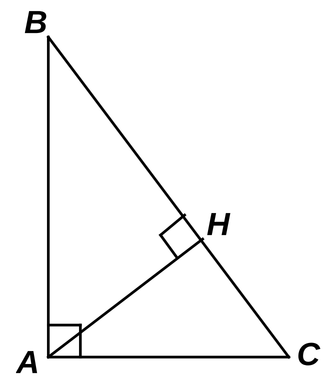
Do *integers* *x*, *y* and *z* exist so that triangle *ABC* has sides of integer length, i.e. *a*, *b* and *c* are also integers?

If yes, then $\left\{\begin{array}{l} \left.\begin{array}{l} x^2+y^2=a^2 \\ y^2+z^2=b^2 \end{array}\right\} \Rightarrow x^2 - z^2 = a^2 - b^2 \\ x^2+z^2=c^2 \end{array}\right. \Rightarrow x^2 = \frac{a^2-b^2+c^2}{2}$.

What do you think?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016 SOLUTION KEY

Team Round



B) 1) $BC^2 = 9^2 + \left(2\sqrt{10}\right)^2 = 121 \Rightarrow BC = 11$

2) In ΔABC, $AB^2 + AC^2 = BC^2$ and $BC = HB + HC = 11$.
3) In ΔAHC, $AH^2 + HC^2 = AC^2$.
4) In ΔAHB, $AH^2 + HB^2 = AB^2$.
Subtracting equation 2) from equation 3),
$HB^2 - HC^2 = (HB + HC)(HB - HC) = AB^2 - AC^2$
$\Rightarrow 11(HB - HC) = AB^2 - AC^2 = 81 - 40 = 41$

Thus, $HB - HC = \underline{\mathbf{\dfrac{41}{11}}}$. Sure beats solving for *HB* and *HC* separately and subtracting!

Alternative solution (Brute Force, ***but*** putting off the inevitable as long as possible):
$BC = 11$ and area(ΔABC) = $\dfrac{1}{2} \cdot 9 \cdot 2\sqrt{10} = \dfrac{1}{2} \cdot 11 \cdot AH \Rightarrow AH = \dfrac{18\sqrt{10}}{11}$ (Let $k = AH^2 = \dfrac{18^2 \cdot 10}{11^2}$.)
From similar triangles, $AH^2 = (HB)(HC) \Rightarrow k = x(11 - x) \Rightarrow x^2 - 11x + k = 0$.
Applying the quadratic formula, $x = \dfrac{11 \pm \sqrt{11^2 - 4k}}{2}$.
Notice the difference we need is simply $\sqrt{11^2 - 4k}$
Substituting, $11^2 - 4k = 11^2 - \dfrac{18^2 \cdot 40}{11^2} = \dfrac{11^4 - 40 \cdot 18^2}{11^2} = \dfrac{14641 - 12960}{11^2} = \dfrac{1681}{11^2} = \dfrac{41^2}{11^2} \Rightarrow \underline{\mathbf{\dfrac{41}{11}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016 SOLUTION KEY

Team Round

C) A typical equation might be $\frac{x+y+z}{3}+v+w=21 \Leftrightarrow 3v+3w+x+y+z=63$

This results in a system of 10 equations in 5 unknowns:

$$\begin{cases} 3v+3w+x+y+z=63 \\ 3v+3x+w+y+z=69 \\ 3v+3y+w+x+z=75 \\ 3v+3z+w+x+y=81 \\ 3w+3x+v+y+z=75 \\ 3w+3y+v+x+z=81 \\ 3w+3z+v+x+y=87 \\ 3x+3y+v+w+z=87 \\ 3x+3z+v+w+y=93 \\ 3y+3z+v+w+x=99 \end{cases}$$

Adding the 10 equations and dividing by 18, we get $v+w+x+y+z=45$.
Subtracting this equation from the first six equations above, we get:

$$\begin{cases} (1)\ v+w=9 \\ (2)\ v+x=12 \\ (3)\ v+y=15 \\ (4)\ v+z=18 \\ (5)\ w+x=15 \\ (6)\ w+y=18 \end{cases}$$

Subtracting (2) – (1), we get $x-w=3$ and solving with (5), $2x=18 \Rightarrow x=9, w=6$, $v=3,\ y=12,\ z=15$

Thus, $(v_1,v_2,v_3,v_4,v_5)=\underline{(\mathbf{3,6,9,12,15})}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016 SOLUTION KEY

Team Round

D) $2x+3 = A(x-2)^2 + B(x-2) + C = A(x^2-4x+4) + Bx - 2B + C$

$$\begin{cases}(1):\ x=0 \Rightarrow 4A-2B+C=3 \\ (2):\ x=1 \Rightarrow 3A-B+C=5 \\ (3):\ x=2 \Rightarrow 4A+C=7\end{cases} \Rightarrow \begin{cases}(1)-(2):\ A-B=-2 \\ (3)-(2):\ A+B=2\end{cases} \Rightarrow A=0, B=2, C=7$$

Thus, $A^2+B^2+C^2 = \underline{\mathbf{53}}$.

E) Since x^2+1 is positive for all values of *x*, the inequality simplifies to
$2\left|x^2-1\right| \le \left(x^2+1\right)+9x+7 = x^2+9x+8 = (x+1)(x+8)$
Since the left side always produces a nonnegative value, the right side of the inequality must also be nonnegative and this is the case for $\boxed{x \le -8 \text{ or } x \ge -1}$.
Values outside this range are extraneous.
To remove the absolute value, we must consider two separate cases.
Case 1: $x \le -1$ or $x \ge 1 \Rightarrow \left|x^2-1\right| = x^2-1$
$2\left(x^2-1\right) \le x^2+9x+8 \Leftrightarrow x^2-9x-10 \le 0 \Leftrightarrow (x-10)(x+1) \le 0 \Rightarrow -1 \le x \le 10$
Some of these values fall outside the domain of definition of this equivalent equation.
The acceptable values are $x=-1$ or $1 \le x \le 10$. [$-1 < x < 1$ are extraneous *for this case*.]

Case 2: $-1 < x < 1 \Rightarrow \left|x^2-1\right| = 1-x^2$
None of these values can be rejected out of hand, since all of these values are a subset of $\boxed{x \le -8 \text{ or } x \ge -1}$. The equivalent inequality is
$2\left(1-x^2\right) \le x^2+9x+8 \Leftrightarrow 3x^2+9x+6 \ge 0 \Leftrightarrow 3\left(x^2+3x+2\right) \ge 0 \Leftrightarrow 3(x+2)(x+1) \ge 0$
$\Rightarrow x \le -2$ or $x \ge -1$. Over the domain of definition for this equivalent equation, we pick up solutions $-1 < x < 1$. Combining the two cases, the required condition is $\underline{\mathbf{-1 \le x \le 10}}$.

Since, for any *positive constant k*, $|x| \le k \Leftrightarrow (-k \le x \le k) \Leftrightarrow (x \ge -k)$ and $(x \le k)$, you might want to complete the following solution with your teammates and/or coach:
$2\left|x^2-1\right| - \left|x^2+1\right| \le 9x+7 \ \Leftrightarrow\ 2\left|x^2-1\right| - \left(x^2+1\right) \le 9x+7 \ \Leftrightarrow\ 2\left|x^2-1\right| \le x^2+9x+8$
$\Leftrightarrow\ -x^2-9x-8 \le 2\left(x^2-1\right) \le x^2+9x+8 \ \Leftrightarrow\ \left(3x^2+9x+6 \ge 0\right)$ **<u>and</u>** $\left(x^2-9x-10 \le 0\right)$

F) The Fibonacci numbers in blocks of 5 are:

<u>1,2,3,5,8</u> <u>13,21 ^ 34,55 ^ 89</u> ^ <u>144 ^ 233 ^ 377 ^ 610, ^ 987</u> ^^ 1597,… .

^ denotes the location of a perfect square.
The perfect cubes are 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, … .
We require that two cubes fit between consecutive Fibonacci numbers.
This first happens for the interval $(987, 1597)$ which contains both 1000 and 1331 $\Rightarrow$
$(j,k) = \underline{(\mathbf{10,15})}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016 ANSWERS

Round 1 Geometry Volumes and Surfaces

A) 150

B) $y = x \pm 1$
(2 answers required)

C) (25920, 3)

Round 2 Pythagorean Relations

A) 25

B) 176

C) $\sqrt{3R}$

Round 3 Linear Equations

A) 20

B) 75 mph

C) $(18,1)$

Round 4 Fraction & Mixed numbers

A) 870 mph

B) 9

C) $\frac{4}{3}$

Round 5 Absolute value & Inequalities

A) 10

B) $-4 \le x \le \frac{3}{2}$

C) $x < -2$ or $-1 < x < 0$ or $1 < x < 2$

Round 6 Evaluations

A) 42

B) 33

C) 8

Team Round

A) 62

B) $\frac{41}{11}$

C) $(3,6,9,12,15)$

D) 53

E) $-1 \le x \le 10$

F) (10, 15)