MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016
ROUND 1 COMPLEX NUMBERS (No Trig)

ANSWERS

A) ______________________________

B) ______________________________

C) (_____________ , _____________)

A) Compute the minimum positive integer A for which $i^A \cdot i^{2A} \cdot i^{4A} \cdot i^{8A} \cdot i^{16A} = i$ is true.

B) Let $z = 0.1 + 6i$. If $a = \dfrac{1}{z + \overline{z}}$ and $bi = z - \overline{z}$, compute $|a + bi|$.

Recall:
If $z = a + bi$, then the conjugate of z (written $\overline{z}$), is defined as $a - bi$.
$|a + bi| = \sqrt{a^2 + b^2}$, the distance from the origin to the point $P(a, b)$ in the complex plane.

C) One of the two possible values of $(3 + 4i)^{\frac{3}{2}}$ may be written as $A + Bi$, where A and B are positive integers. Compute the ordered pair (A, B).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016 SOLUTION KEY

Round 1

A) Solution #1:

Since $i^4 = 1$, $\left(i^4\right)^{\text{any integer power}}$ equals 1.

Therefore, $i^A \cdot i^{2A} \cdot i^{4A} \cdot i^{8A} \cdot i^{16A} = i^A \cdot i^{2A} \cdot \cancel{i^{4A}} \cdot \cancel{i^{8A}} \cdot \cancel{i^{16A}} = i^{3A} = i^1 = i^5 = i^9$ and we find the minimum value of A by equating exponents. $3A = 9 \Rightarrow A = \underline{\mathbf{3}}$.

Solution #2:

$i^A \cdot i^{2A} \cdot i^{4A} \cdot i^{8A} \cdot i^{16A} = i^{31A}$

$A = 1 \Rightarrow i^{31} = \left(i^4\right)^7 i^3 = 1^7(-i) = -i$

$A = 2 \Rightarrow i^{62} = \left(i^4\right)^{15} i^2 = -1$

$A = 3 \Rightarrow i^{93} = \left(i^4\right)^{23} i = i \Rightarrow A_{\min} = \underline{\mathbf{3}}$.

B) $z = 0.1 + 6i \Rightarrow \overline{z} = 0.1 - 6i$. If $a = \dfrac{1}{z + \overline{z}}$ and $bi = z - \overline{z}$,

$a = \dfrac{1}{.2} = 5$, $b = 12 \Rightarrow \left|5 + 12i\right| = \sqrt{5^2 + 12^2} = \sqrt{169} = \underline{\mathbf{13}}$ (or recall 5-12-13 Pythagorean Triple)

C) $\left(3 + 4i\right)^{\frac{1}{2}} = C + Di \Rightarrow (C + Di)^2 = 3 + 4i \Rightarrow \begin{cases} C^2 - D^2 = 3 \\ 2CD = 4 \end{cases} \Rightarrow (C, D) = (2, 1)$ or $\left(-2, -1\right)$.

$\left(3 + 4i\right)^{\frac{3}{2}} = \left(\left(3 + 4i\right)^{\frac{1}{2}}\right)^3 = (2 + i)^3 = (2 + i)^2(2 + i) = (3 + 4i)(2 + i) = 2 + 11i$

Thus, $\left(A, B\right) = \underline{\mathbf{(2, 11)}}$.

$\left(-2, -1\right)$ is rejected, since $(-2 - i)^3 = \left(-1\right)^3(2 + i)^3 = -2 - 11i$ and it was required that A and B be positive integers.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016
ROUND 2 ALGEBRA 1: ANYTHING

ANSWERS

A) (______________ , ______________)

B) ________________________________

C) ________________________________

A) Richard and Anne were the only candidates running for President of the Chemistry Club. There were no write-in votes for other candidates, so all of the votes cast were for either Richard or Anne.
If 7 of the votes that went to Anne had gone to Richard instead, the candidates would have tied.
If 5 of the votes Richard received had gone to Anne,
she would have had 7 times as many votes as Richard.
How many votes did each candidate receive?
Express your answer as an ordered pair (Anne, Richard).

B) Given: $x \blacklozenge y = x^2 + y^2 + xy$
If $2 \blacklozenge x = 12$, compute <u>all</u> possible values of $3 \blacklozenge x$.

C) The force F exerted between two bodies varies inversely as the square of the distance (d) between the two bodies and jointly as the masses of the two bodies (m_1 and m_2).
$F = 1200$, when $(m_1, m_2, d) = (5, 20, 8)$.
If the distance between the masses remains unchanged, but each mass is <u>increased</u> by n, the resulting force F is 2208. Compute n.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016 SOLUTION KEY

Round 2

A) $A-7=R+7 \Rightarrow A=R+14$

$A+5=7(R-5) \Leftrightarrow R+19=7R-35 \Rightarrow 6R=54 \Rightarrow R=9 \Rightarrow \underline{(\mathbf{23,9})}$.

B) Given: $x \blacklozenge y = x^2+y^2+xy$

$2 \blacklozenge x = 4+x^2+2x=12 \Leftrightarrow x^2+2x-8=(x+4)(x-2)=0 \Rightarrow x=-4,2$.

$3 \blacklozenge (-4) = 9+16-12 = \underline{\mathbf{13}}$.

$3 \blacklozenge 2 = 9+4+6 = \underline{\mathbf{19}}$.

C) Given: $F = 1200$, when $(m_1, m_2, d) = (5, 20, 8)$

$F = k\dfrac{m_1 m_2}{d^2} \Rightarrow 1200 = \dfrac{100k}{64} \Rightarrow k = 12(64)$ Then:

$2208 = 12(64)\dfrac{(5+n)(20+n)}{64} \Rightarrow 100+25n+n^2 = \dfrac{2208}{12} = 184$

$\Rightarrow n^2+25n-84=(n-3)(n+28)=0 \Rightarrow n = \underline{\mathbf{3}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016
ROUND 3 PLANE GEOMETRY: AREAS OF RECTILINEAR FIGURES

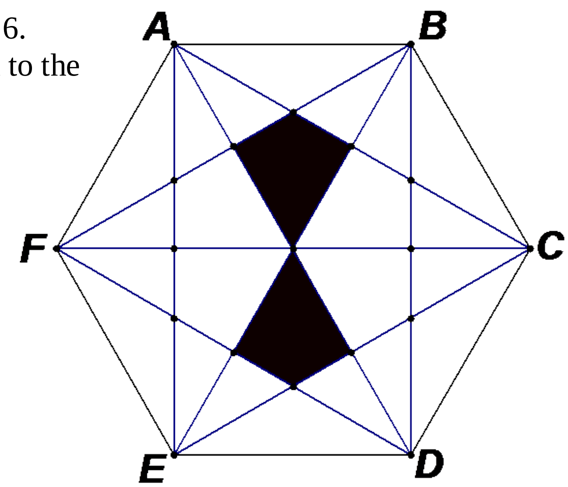
ANSWERS

A) _______________ : _______________

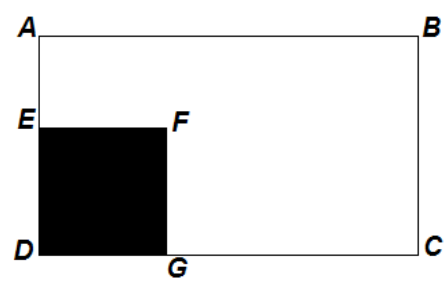
B) ________________________________

C) ________________________________

A) *ABCDEF* is a regular hexagon with side of length 6. Compute the ratio of the area of the shaded region to the area of the unshaded region in hexagon *ABCDEF*.



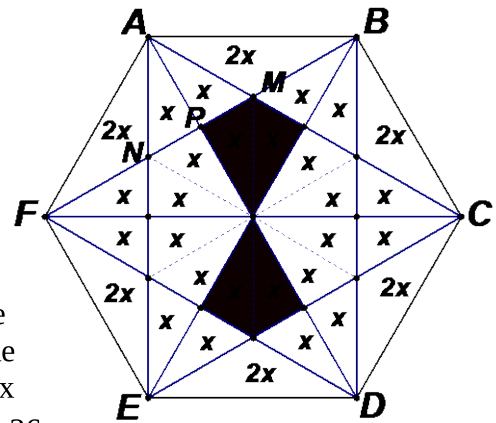
B) *DEFG* is a square of side *x*. *ABCD* is a rectangle with sides $AB = 18$ and $BC = 14$.
The ratio of the area of the unshaded region to the area of the shaded region equals $\frac{AB}{BC}$. Compute *x*.



C) In a rhombus with side 7, the long diagonal has length 11. Compute the length of the altitude of this rhombus.

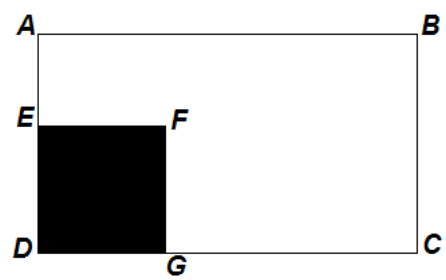
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016 SOLUTION KEY

Round 3



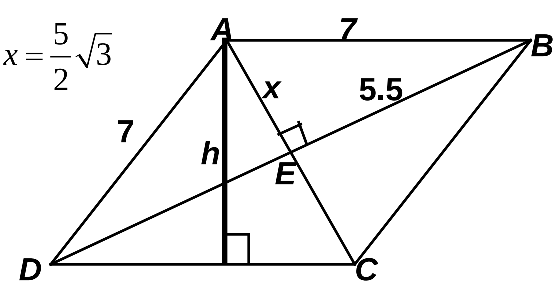
A) Clearly, the regions marked with x's are congruent and congruent regions have the same area. Therefore, let x denote the <u>area</u> of each of these regions. ΔFAN and ΔMAN are not congruent, but they do have the same area. ($FN = NM$ and $\overline{AP}$ is a common altitude when these sides are taken to be the bases of the two triangles.) The area of hexagon $ABCDEF$ equals the sum of an inner hexagon, six equilateral triangles and six congruent obtuse triangles, namely, $12x + 12x + 6(2x) = 36x$.
The required ratio is 4 : (36 -- 4) = <u>**1 : 8**</u>.

B) Given: $DEFG$ is a square of side x. $ABCD$ is a rectangle with sides $AB = 18$ and $BC = 14$.



$$\frac{18\cdot 14 - x^2}{x^2} = \frac{9}{7} \Rightarrow 7\cdot 18\cdot 14 - 7x^2 = 9x^2$$
$$\Rightarrow x^2 = \frac{7\cdot 18\cdot 14}{16} = \frac{4\cdot 9\cdot 49}{16}$$
$$\Rightarrow x = \frac{3\cdot 7}{2} = \underline{\mathbf{\frac{21}{2}}} \text{ or } (\underline{\mathbf{10.5}}).$$

C) Since the diagonals of a rhombus are perpendicular,



we have $x^2 + (5.5)^2 = 7^2 \Rightarrow x^2 = 49 - 30.25 = 18\frac{3}{4} = \frac{75}{4} \Rightarrow x = \frac{5}{2}\sqrt{3}$
Thus, the <u>short</u> diagonal has length $5\sqrt{3}$.
Note that $5\sqrt{3} < 11$ (since $5^2 \cdot 3 < 11^2$).
Invoking the area formulas for any rhombus, we have
$$7h = \frac{1}{2}\cdot 11\cdot 5\sqrt{3} \Rightarrow h = \underline{\mathbf{\frac{55\sqrt{3}}{14}}}.$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016
ROUND 4 ALG 1: FACTORING AND ITS APPLICATIONS

ANSWERS

A) ______________________________

B) (_____________ , _____________)

C) ______________________________

A) Compute all values of x for which $(3x-2)(3x+1)=4$.

B) Given: $A > B > 0$ and $A \cdot B = 180$, for integers A and B.
The greatest common factor of A and B is 1 for exactly j distinct ordered pairs (A, B) and greater than 1 for exactly k distinct ordered pairs (A, B). Compute the ordered pair (j, k).

C) For $A > 0$, $\dfrac{(x+2)^2-81}{(7-x)(A+x)} \geq 0$ is satisfied for exactly 3 distinct integer values of x.
Compute all possible integer values of A.

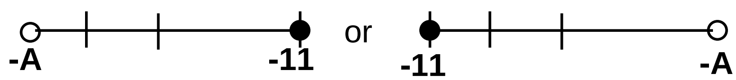
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016 SOLUTION KEY

Round 4

A) $(3x-2)(3x+1)=4 \Leftrightarrow 9x^2-3x-6=0 \Leftrightarrow 3(3x^2-x-2)=3(3x+2)(x-1)=0 \Rightarrow x=$ $\underline{\mathbf{-\frac{2}{3}, 1}}$

B) Examining the factorization of $180=2^2\cdot 3^2\cdot 5^1$, we see 180 has 18 positive factors which will form 9 ordered pairs (*A*, *B*) where $A\cdot B=180$ and $A>B>0$.
GCF = 1: (180, 1), (45, 4), (36, 5), (20, 9)
GCF = 2: (90, 2), (18, 10)
GCF = 3: (60, 3), (15, 12)
GCF = 6: (30, 6)
Thus, $(j,k)=\underline{\mathbf{(4,5)}}$.

C) $\dfrac{(x+2)^2-81}{(7-x)(A+x)}\ge 0 \Leftrightarrow \dfrac{x^2+4x-77}{(7-x)(A+x)}\ge 0 \Leftrightarrow \dfrac{(x+11)(x-7)}{(7-x)(A+x)}\ge 0 \Leftrightarrow \dfrac{-x-11}{A+x}\ge 0$, provided $x\ne 7$.
or, equivalently, $\dfrac{x+11}{A+x}\le 0$. The solution set is between -11 and -*A*.
However, we must consider two cases: -*A* to the left of -11 and -*A* to the right of -11



To guarantee exactly 3 integer solutions, *A* = **<u>8</u>** (-11, -10, -9) or **<u>14</u>** (-13, -12, -11).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016
ROUND 5 TRIG: FUNCTIONS OF SPECIAL ANGLES

ANSWERS

A) ______________________________

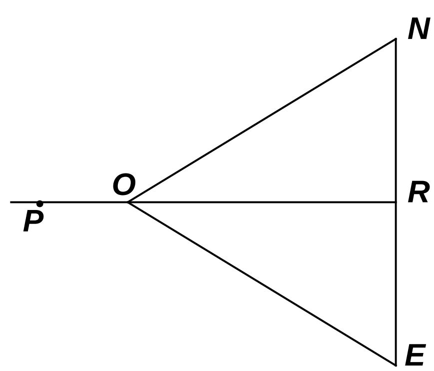
B) ______________________________

C) ______________________________

A) Determine the minimum value of A for which $\sin A = \frac{1}{2}$ and $A > 800°$.

B) Compute $\dfrac{\tan 60° - \sin 270°}{\sin 210° + \cos 330° - \tan(-225°)}$

C) ΔEON is equilateral and R is the midpoint of $\overline{NE}$.
P, O and R are collinear.
If $\overrightarrow{OS}$ bisects $\angle NOR$. Compute $\tan(POS)$.



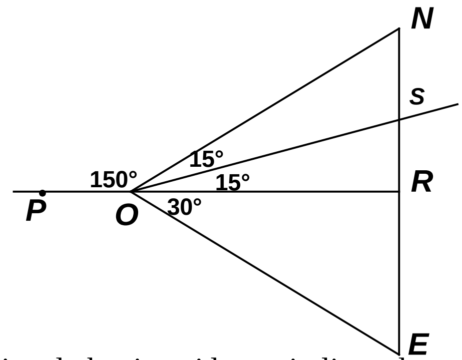
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016 SOLUTION KEY

Round 5

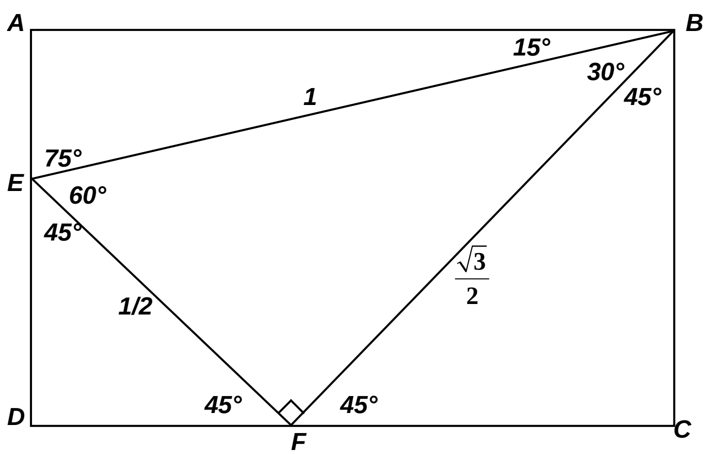
A) $A = \begin{cases} (1)\ \ 30° + n(360°) \\ (2)\ \ 150° + n(360°) \end{cases}$

For (1), the minimum value of n is 3, producing 1110
For (2), the minimum value of n is 2, producing **870**.

B) $\dfrac{\tan 60° - \sin 270°}{\sin 210° + \cos 330° - \tan(-225°)} = \dfrac{\sqrt{3} - (-1)}{-\frac{1}{2} + \frac{\sqrt{3}}{2} - (-1)} = \dfrac{\sqrt{3}+1}{\frac{\sqrt{3}+1}{2}} = \mathbf{2}$



C) Clearly, $m\angle POS = 165$.
The tangent of an obtuse angle is negative.
Since $\tan\theta = -\tan(180-\theta)$, $\tan 165° = -\tan 15°$.
Solution #1: Using only special angles (30°, 45° and 60°)
Consider rectangle *ABCD* with an embedded 30 – 60 – 90 right triangle having sides as indicated.



1) $FC = BC = AD = \dfrac{\sqrt{3}}{2}\cdot\dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{3}}{2\sqrt{2}} = \dfrac{\sqrt{6}}{4}$ 2) $DE = DF = \dfrac{1}{2}\cdot\dfrac{1}{\sqrt{2}} = \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}$

3) $AB = DF + FC = \dfrac{\sqrt{2}}{4} + \dfrac{\sqrt{6}}{4} = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ 4) $AE = AD - DE = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

5) $\tan 15° = \dfrac{AE}{AB} = \dfrac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \dfrac{\left(\sqrt{6}-\sqrt{2}\right)^2}{6-2} = \dfrac{8-2\sqrt{12}}{4} = 2-\sqrt{3} \Rightarrow \tan 165° = \underline{\mathbf{\sqrt{3}-2}}$

Solution #2: (BORING expansion formulas!)

$\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A\tan B} \Rightarrow \tan 15° = \dfrac{\tan 60° - \tan 45°}{1+\tan 60°\tan 45°} = \dfrac{\sqrt{3}-1}{1+\sqrt{3}} = \text{etc.}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016
ROUND 6 PLANE GEOMETRY: ANGLES, TRIANGLES AND PARALLELS

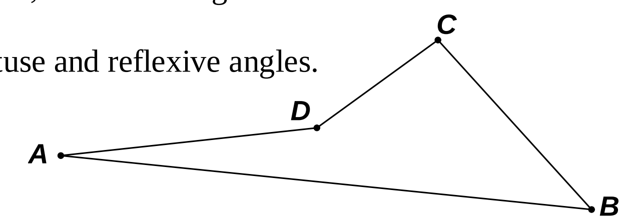
ANSWERS

A) ______________________________

B) (_____________ , _____________)

C) ______________________________

A) A concave quadrilateral has interior angles which measure $x°$, $(2x)°$, $(4x)°$ and $(8x)°$.
One of these angles is obtuse, and another is reflexive, i.e. with a degree measure between 180° and 360°.
Compute the <u>sum</u> of the degree-measures of the obtuse and reflexive angles.

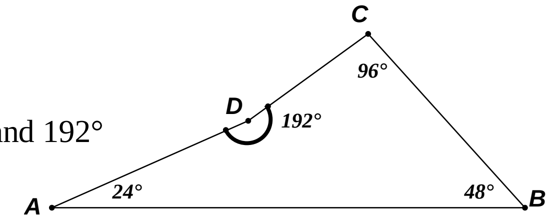


B) Each interior angle of a regular polygon with N sides measures more than 178°and less than 179°. The minimum value of N is m and the maximum value of N is M.
Compute the ordered pair (m, M).

C) Shortly after 5PM, the hour and minute hands of a circular clock form an angle of 110°. Within the hour, this happens again. Compute the elapsed time (in minutes) between these two occurrences.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016 SOLUTION KEY

Round 6



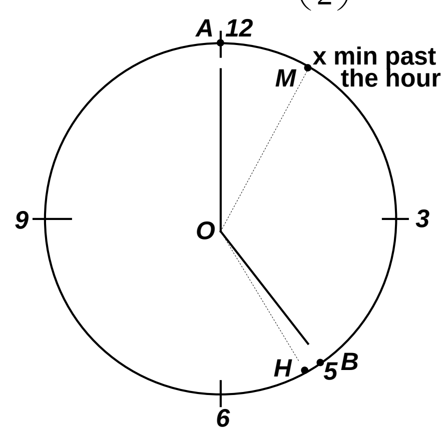
A) $x+2x+4x+8x=360 \Leftrightarrow 15x=360 \Rightarrow x=24$.
Thus, the 4 angle measures are 24°, 48°, 96° (obtuse) and 192° (reflexive).
The required sum is **288**.

B) $178<\frac{180(n-2)}{n}<179 \Rightarrow 178n<180n-360$ and $180n-360<179n$
$\Rightarrow n>180$ and $n<360 \Rightarrow (m,M)=(\mathbf{181,359})$.

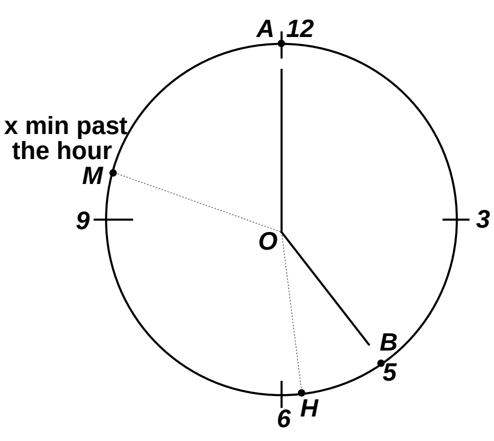
C) The minute hand moves 12 times as fast as the hour hand.
In one hour, the minute hand makes a complete revolution, i.e. turns through an angle of 360° or, equivalently, 6° every minute. The hour hand turns through $\frac{360°}{12}=30°$ every hour, or, equivalently, $\frac{1}{2}^{\circ}$ every minute.

In *x* minutes, the minute hand turns through $(6x)^{\circ}$ and the hour hand turns through $\left(\frac{x}{2}\right)^{\circ}$.

Assume the first 110° angle occurs at *x* minutes past 5:00, i.e. $m\angle MOH=110°$
At 5:00, $m\angle AOB=5(30°)=150°$
At *x* minutes past 5:00, $m\angle AOM=(6x)^{\circ}$ and $m\angle BOH=\left(\frac{x}{2}\right)^{\circ}$.
Equating two different expressions for $\angle AOH$,
$6x+110=150+\frac{x}{2} \Rightarrow x=\frac{40}{11}$.



For the second occurrence the diagram looks like the clock below:



Equating two different expressions for $\angle AOM$,
$6x=150+\frac{x}{2}+110 \Rightarrow x=\frac{520}{11}$
The difference $\frac{520-80}{11}=\frac{440}{11}=\mathbf{40}$ minutes.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016
ROUND 7 TEAM QUESTIONS

ANSWERS

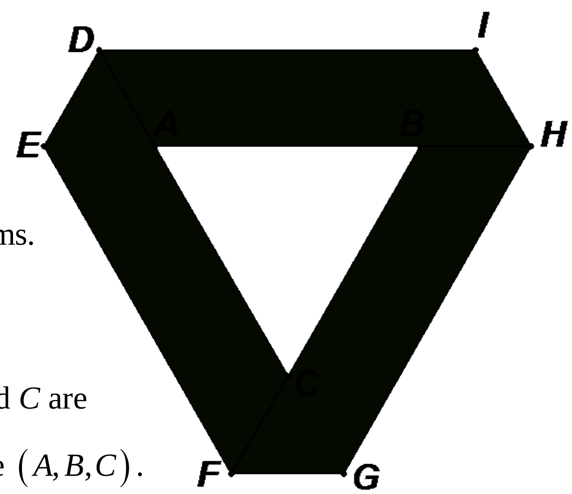
A) ____________________________ D) (___ , ___ , ___ , ___ , ___ , ___)

B) (__ , __) S = { ___________ } E) ____________________________

C) (______ , ______ , ______) F) ____________________________

A) Given: $\sum_{n=1}^{n=32}(1+i)^n = k(1-i)$ Compute k.

B) There are K possible digit-sums for the set of 2-digit primes.
Let S be the set of the most frequently occurring digit-sums and N be the number of times each of these sums occurred.
Determine the ordered pair (K,N) and the set S.
Recall: The digit-sum of a number is the sum of the digits in that number.



C) ABC is an equilateral triangle with area 1 unit^2.
$DEFC$, $FGHB$ and $HIDA$ are congruent parallelograms.
If $FG = \frac{1}{2}$, the area of the shaded region can be expressed in the form $\frac{A\left(\sqrt{3}+B\sqrt[4]{3}\right)}{C}$, where A, B and C are relatively prime integers. Compute the ordered triple (A,B,C).

D) $36x^2 - 3xy - 60y^2 + 18x + 38y - 4$ factors as the product of two trinomials, namely $Ax + By + C$ and $Dx + Ey + F$, where each constant is an integer.
If $AB < 0$, compute the ordered 6-tuple of constants (A,B,C,D,E,F).

E) Compute $\sin 10° \sin 30° \sin 50° \sin 70°$.

F) The exterior angles (one at each vertex) of ΔABC measure $\left(2(x+y)+8\right)°, (5y-x)°$ and $(3x+y-44)°$, where x and y are integers.
$x+y>41$ and $2y-x>46$
The obtuse angle formed by the bisectors of the acute interior angles of ΔABC measures 146°.
Compute the degree-measure of the <u>largest</u> <u>exterior</u> angle of ΔABC.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016 SOLUTION KEY

Team Round

A) The summation $\sum_{n=1}^{n=32}(1+i)^n$ consists of 32 terms.

Since powers of i cycle in blocks of 4, consider any series of 4 consecutive terms.
Consider the simplest 4-block $(1+i)^0+(1+i)^1+(1+i)^2+(1+i)^3$.

This simplifies to $1+(1+i)+2i+2i(1+i)=2+3i+2i-2=5i$. Thus, expressions of the form $(1+i)^k+(1+i)^{k+1}+(1+i)^{k+2}+(1+i)^{k+3}$ simplify to $(1+i)^k\cdot 5i$, for <u>any</u> integer *k*.

The index for the given summation starts at 1.

$k=1 \Rightarrow (1+i)^1\cdot(5i)=5i+5i^2=-5(1-i)$

For the given summation, *k* = 1, 5, 9, … , 29 generate 32 terms, 8 blocks of 4 terms each.
With a little effort, verify that the 4-block sums for *k* = 5, 9 and 13 are $20(1-i)$, $-80(1-i)$ and $320(1-i)$. The coefficients of the common binomial term form a geometric sequence with a common multiplier of –4. We must sum 8 terms from this sequence.

The required sum is $\frac{a(1-r^n)}{1-r}(1-i)=\frac{-5\left(1-(-4)^8\right)}{1-(-4)}(1-i)$.

$\Rightarrow k=\frac{5(4^8-1)}{5}=4^8-1=2^{16}-1=2^{10}\cdot 2^6-1=1024\cdot 64-1=$ <u>**65535**</u>.

B) The 2-digit primes are: 11, 13, 17, 19, 23, 29, 31
37, 41, 43, 47, 53, 59, 61
67, 71, 73, 79, 83, 89, 97

There are 21 2-digit primes.
The following chart summarizes the possible digit-sums and their corresponding frequencies

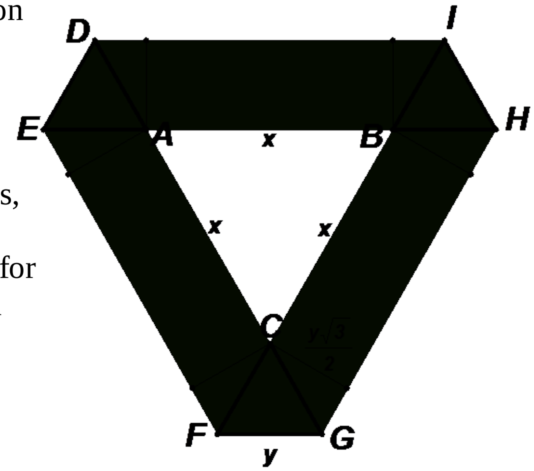
2	4	5	7	8	10	11	13	14	16	17
1	2	2	2	3	3	3	1	1	2	1
11	13 31	23 41	43 61	17 53 71	19 37 73	29 47 83	67	59	79 97	89

The fact that the 11 frequencies add up to 21 is a double check.
Thus, $(K,N)=$ <u>$(\mathbf{11,3})$</u>, $S=$ <u>$\{\mathbf{8,10,11}\}$</u>. (The elements in the set *S* may be listed in any order.)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016 SOLUTION KEY

Team Round

C) From the diagram at the right, we see that the shaded region is comprised of 3 congruent equilateral triangles (*ADE*, *BHI* and *CFG*), 6 congruent 30-60-90 right triangles and 3 congruent rectangles. Any pair of 30-60-90 right triangles can be combined to form an equilateral triangle congruent to the named equilateral triangles. Thus, the area of the shaded region is equivalent to 6 equilateral triangles and 3 rectangles. The numbers are atrocious, so for now, forget the numerical values and assume $AB = x$ and $FG = y$. The required area is



$$6\left(\frac{y^2\sqrt{3}}{4}\right)+3\left(x\cdot\frac{y\sqrt{3}}{2}\right)=\boxed{\frac{3\sqrt{3}}{2}\left(y^2+xy\right)}$$

Now we find *x*, substitute and simplify.

$$\frac{x^2\sqrt{3}}{4}=1\Rightarrow x^2=\frac{4}{\sqrt{3}}=\frac{4\sqrt{3}}{3}=\frac{4\cdot 3\sqrt{3}}{9}=\frac{4\sqrt{27}}{9}\Rightarrow x=\frac{2}{3}\sqrt[4]{27}.$$

For $y=\frac{1}{2}$, the required area is $\frac{3\sqrt{3}}{2}\left(\frac{1}{4}+\frac{x}{2}\right)=\frac{3\sqrt{3}}{8}(1+2x)=\frac{3\sqrt{3}}{8}\left(1+\frac{4\sqrt[4]{27}}{3}\right)$

$$=\frac{\sqrt{3}}{8}\left(3+4\sqrt[4]{27}\right)=\frac{3\sqrt{3}+4\sqrt{3}\cdot\sqrt[4]{27}}{8}=\frac{3\sqrt{3}+4\sqrt[4]{9}\cdot\sqrt[4]{27}}{8}=\frac{3\sqrt{3}+4\sqrt[4]{3^5}}{8}=\frac{3\left(\sqrt{3}+4\sqrt[4]{3}\right)}{8}\Rightarrow \underline{(\mathbf{3,4,8})}.$$

D) Solution #1: Quadratic Formula

But the QF works for an equation in a single variable??? Treat *y* as a constant!

$$36x^2-3xy-60y^2+18x+38y-4=36x^2+(18-3y)x-\left(60y^2-38y+4\right)$$

$$x=\frac{3y-18\pm\sqrt{(18-3y)^2+4(36)\left(60y^2-38y+4\right)}}{72}.$$ Factoring a 9 out of the radicand allows us to eliminate a factor of 3 in the numerator and denominator.

$$\boxed{x=\frac{y-6\pm\sqrt{(6-y)^2+16\left(60y^2-38y+4\right)}}{24}}.$$ Expanding the radicand, we are looking for a perfect square trinomial.

$$(6-y)^2+16\left(60y^2-38y+4\right)$$
$$36-12y+y^2+960y^2-608y+64=961y^2-620y+100$$

Voila! Both the lead coefficient and the constant term are perfect squares. We have $(31y-10)^2$.

Simplifying the boxed equation, $x=\frac{y-6\pm(31y-10)}{24}=\frac{32y-16}{24},\ \frac{-30y+4}{24}$.

Reducing the fractions, $x=\frac{4y-2}{3},\ \frac{-15y+2}{12}$. Clearing the fractions and transposing terms,

$3x-4y+2=0$, $12x+15y-2=0$, and these are our two factors $\Rightarrow$ $\underline{\mathbf{(3,\ -4,\ 2,\ 12,\ 15,\ -2)}}$.

Team Round

D) continued

Solution #2: Indeterminate Coefficients (or Systematic Guess and Check)

Key Concept: Parity: Even + Odd = Odd / Even x Odd = Even, etc.
Signs ($\pm$) are not so important, since interchanging positive and negative factors in a product maintains the negative result.
Matching the coefficients of

$$\left(Ax+By+C\right)\left(Dx+Ey+F\right)=ADx^2-\left(AE+BD\right)xy+BEy^2+\left(AF+DC\right)x+\left(BF+CE\right)y+CF$$

with the coefficients of $36x^2-3xy-60y^2+18x+38y-4$, we get an *exciting* system of 6 equations in the 6 unknown constants.

$$\begin{cases}(1)\ x^2 & AD=36\\ (2)\ xy & AE+BD=-3\\ (3)\ y^2 & BE=-60\\ (4)\ x & AF+DC=18\\ (5)\ y & BF+CE=38\\ (6) & CF=-4\end{cases}$$

There are lots of possibilities. To minimize the guesswork, we zero in on equation #2 (the only one with an *odd* sum) and #6 (*fewest* number of factors) and start "guessing".

If $C=1$ and $F=4$, then $\begin{cases}4A+D=18\\4B+E=38\end{cases}\Rightarrow$ both D and E are even and this contradicts equation #2, since the sum $AE+BD$ is supposed to be odd.

Therefore, we definitely know that $C=2,\ F=-2$ (**or vice versa**).

Equations #3, 5 $\Rightarrow\begin{cases}BE=-60\\ \boxed{B-E=19}\end{cases}\Rightarrow\left(B,E\right)=\left(-15,4\right)$

Equations #1, 2 $\Rightarrow\begin{cases}\boxed{-4A+15D=-3}\\ AD=36\end{cases}\Rightarrow\left(A,D\right)=\left(12,3\right)$

Checking in equation #4, $12\cdot\left(-2\right)+3\cdot2=-18$ Oops! It must have been $\left(C,F\right)=\left(-2,2\right)$

Voila! The factors are $\left(12x+15y-2\right)$ and $\left(3x-4y+2\right)$.

If $AB<0$, the required 6-tuple is $\underline{\mathbf{\left(3,-4,2,12,15,-2\right)}}$.

<u>Challenge:</u> If the question had asked for $AB+AE+AF+BD+BE+BF+CD+CE+CF$, it would have been MUCH easier. Why?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016 SOLUTION KEY

Team Round

E) Knowing the expansions for sin(A ± B) and cos(A ± B), we see that

$$\begin{cases} (1)\ \cos(A-B)+\cos(A+B)=2\cos A\cos B \\ (2)\ \cos(A-B)-\cos(A+B)=2\sin A\sin B \\ (3)\ \sin(A+B)+\sin(A-B)=2\sin A\cos B \\ (4)\ \sin(A+B)-\sin(A-B)=2\cos A\sin B \end{cases}$$

$\sin 10°\sin 30°\sin 50°\sin 70° =$

Regrouping for implementation of rule #2: $\frac{1}{4}\left(2\sin 10°\sin 70°\right)\left(2\sin 30°\sin 50°\right)$

Applying rule #2: $\frac{1}{4}\left(\cos 60°-\cos 80°\right)\left(\cos 20°-\cos 80°\right)$

FOILing: $\frac{1}{4}\left(\cos 60°\cos 20°-\cos 60°\cos 80°-\cos 80°\cos 20°+\cos 80°\cos 80°\right)$

Regrouping: $\frac{1}{4}\cdot\frac{1}{2}\left(2\cos 60°\cos 20°-2\cos 60°\cos 80°-2\cos 80°\cos 20°+2\cos 80°\cos 80°\right)$

Applying rule #1 (to each of the 4 products):

$$\frac{1}{8}\left(\left(\cos 40°+\cos 80°\right)-\left(\cos(-20°)+\cos 140°\right)-\left(\cos 60°+\cos 100°\right)+\left(\cos 0°+\cos 160°\right)\right)$$

$$\frac{1}{8}\left(\left(\cos 40°+\cos 80°\right)-\left(\cos 20°-\cos 40°\right)-\left(\cos 60°-\cos 80°\right)+\left(1-\cos 20°\right)\right)$$

$$\frac{1}{8}\left(2\cos 40°+2\cos 80°-2\cos 20°-\frac{1}{2}+1\right)$$

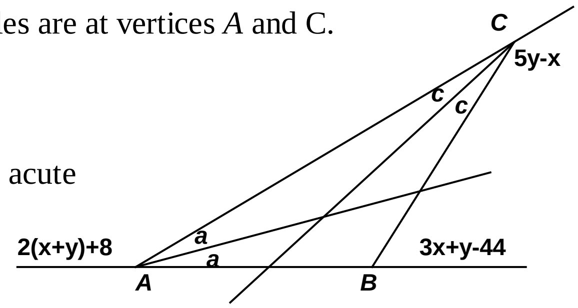
$$\frac{1}{8}\left(2(\cos 40°+\cos 80°)-2\cos 20°-\frac{1}{2}+1\right)$$

Now applying rule #1 in reverse, $A - B = 40$ and $A + B = 80 \Rightarrow (A, B) = (60, 20)$.

$$\frac{1}{8}\left(2(2\cos 60°\cos 20°-\cos 20°)-\frac{1}{2}+1\right)=\frac{1}{8}\left(2(\cos 20°-\cos 20°)-\frac{1}{2}+1\right)=\frac{1}{8}\left(\frac{1}{2}\right)=\underline{\mathbf{\frac{1}{16}}}$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016 SOLUTION KEY

Team Round



F) From the diagram it *appears* that the acute interior angles are at vertices A and C.
But is this the case?

$x+y>41 \Rightarrow 2(x+y)+8>90 \Rightarrow \angle CAB$ is acute

$\begin{cases} x+y>41 \\ 2(2y-x>46) \end{cases} \Rightarrow 5y-x>92+41=133 \Rightarrow \angle BCA$ is acute

Thus, the interior angles at A and C are in fact acute!

$a+c+146=180 \Rightarrow a+c=34$

$\Rightarrow 2a+2c=68 \Rightarrow 2(x+y)+8+(5y-x)=360-68 \Rightarrow x+7y=284$

The exterior angles (one at each vertex) must total 360°.

Therefore, $(2(x+y)+8)+(5y-x)+(3x+y-44)=360 \Rightarrow x+2y=99$

Subtracting, $5y=185 \Rightarrow y=37,\ x=25$.

The exterior angles are 132, 68 and **<u>160</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016 ANSWERS

Round 1 Algebra 2: Complex Numbers (No Trig)

A) 3 B) 13 C) (2, 11)

Round 2 Algebra 1: Anything

A) (23, 9) B) 13, 19 (in any order) C) 3

Round 3 Plane Geometry: Area of Rectilinear Figures

A) 1 : 8 B) 10.5 (or $\frac{21}{2}$) C) $\frac{55\sqrt{3}}{14}$

Round 4 Algebra: Factoring and its Applications

A) $-\frac{2}{3}, 1$ B) (4, 5) C) 8, 14

Round 5 Trig: Functions of Special Angles

A) 870 B) 2 C) $\sqrt{3}-2$

Round 6 Plane Geometry: Angles, Triangles and Parallels

A) 288 B) $(181, 359)$ C) 40

Team Round

A) 65535

B) $(11,3)$, $S=\{8,10,11\}$

C) (3, 4, 8)

D) $(3,-4,2,12,15,-2)$

E) $\frac{1}{16}$

F) 160