MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2016
ROUND 1 TRIG: RIGHT ANGLE PROBLEMS, LAWS OF SINES AND COSINES

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) The short leg in right triangle *ABC* has length 16. The hypotenuse is 2 units longer than the long leg. Compute the area of ΔABC.

B) In ΔABC, $AB = 12$, $BC = 15$, and $AC = 8$. Compute $\dfrac{\sin B + \sin C}{\sin A}$.

C) In right triangle *ABC*, $m\angle C = 90^\circ$, median $AN = 2\sqrt{2}$, and median $BP = 3\sqrt{3}$. Compute the length of median $\overline{CM}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2016 SOLUTION KEY

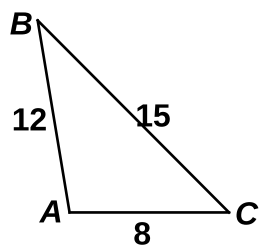
Round 1

A) Let the hypotenuse and long leg have lengths (x + 2) and x.

Then: $16^2 + x^2 = (x+2)^2 \Leftrightarrow 256 + x^2 = x^2 + 4x + 4 \Rightarrow 64 = x + 1 \Rightarrow x = 63$

Thus, the area is $\frac{1}{2}\cdot 16\cdot 63 = 8\cdot 63 = \underline{\mathbf{504}}$.

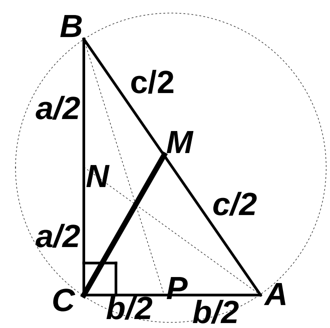
B) According to the Law of Sines,



$$\frac{\sin A}{15} = \frac{\sin B}{8} = \frac{\sin C}{12} = n \Rightarrow \begin{cases} \sin A = 15n \\ \sin B = 8n \\ \sin C = 12n \end{cases}$$

Therefore, $\frac{\sin B + \sin C}{\sin A} = \frac{8n + 12n}{15n} = \frac{20}{15} = \underline{\mathbf{\frac{4}{3}}}$.

C) In right triangles BCP and ACN,



$$\begin{cases} a^2 + \left(\frac{b}{2}\right)^2 = 8 \Rightarrow 4a^2 + b^2 = 32 \\ b^2 + \left(\frac{a}{2}\right)^2 = 27 \Rightarrow a^2 + 4b^2 = 108 \end{cases} \Rightarrow a^2 + b^2 = \frac{140}{5} = 28$$

But $a^2 + b^2 = c^2 = 28 \Rightarrow c = 2\sqrt{7} \Rightarrow CM = \underline{\sqrt{7}}$.

FYI:
The midpoint of the hypotenuse is the center of the circumscribed circle, i.e. the circle which passes through the 3 vertices of the right triangle *ABC*.
The medians in ANY triangle are concurrent, i.e. pass through a common point.
The point of concurrency divides each median into segments whose lengths are in a 2 : 1 ratio.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2016
ROUND 2 ARITHMETIC/NUMBER THEORY

ANSWERS

A) (______________ , ______________)

B) ________________________________

C) ________________________________

A) The value of $n!$ gets large very quickly, but the sum of the digits of $n!$ increases slowly. Let P = minimum value of n for which Q, the sum of the digits of $n!$, exceeds 10. Compute the ordered pair (P, Q).

Note: $n!$ (read n factorial) is defined as the product $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$.

B) Find the remainder when 7^{355} is divided by 4.

C) A two-digit positive integer N leaves a remainder of 1 when divided by 5. If the digits are reversed, this new integer leaves a remainder of 3 when divided by 5. What is the remainder when the sum of all integers N satisfying these conditions is divided by 9?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2016 SOLUTION KEY

Round 2

A) Simply build a table of n!-values.

n	$n!$	Digitsum	n	$n!$	Digitsum
2	2	2	7	5040	9
3	6	6	8	40320	9
4	24	6	**9**	362880	**27**
5	120	3	10		
6	720	9			

Thus, $(P,Q) = \underline{(\mathbf{9,27})}$

B) Looking for a pattern: $7^1 = 4\cdot 1 + \boxed{3}$, $7^2 = 49 = 4\cdot 12 + \boxed{1}$, $7^3 = 343 = 4\cdot 85 + \boxed{3}$, $7^4 = 2401 = 4\cdot 600 + \boxed{1}$, $7^5 = 16807 = 4\cdot 4201 + \boxed{3}, \ldots$

This *suggests* that the remainders alternate between 3 and 1 and that the required remainder is **3**, since the exponent 355 is odd.

This can be summarized as $7^{\text{odd}} \equiv 3 \pmod 4$ and $7^{\text{even}} \equiv 1 \pmod 4$, where (mod n) denotes the remainder upon division by n and $\equiv$ is read "is congruent to".

Does this alternating pattern really continue? Removing any doubt .....

Consider that $7^n = (4+3)^n$. Each term in the expansion will contain a factor of 4, except the last term 3^n, so we must examine powers of 3 to determine the remainder.

$3^n = (2+1)^n$ and the last terms in the expansion will be $2n+1$. If n is even, then this is 1 more than a multiple of 4 $\left(n = 2k \text{ (i.e., even)} \Rightarrow 2n+1 = 2(2k)+1 = 4k + \boxed{1}\right)$; if n is odd, this is 3 more than a multiple of 4 $\left(n = 2k+1 \text{ (i.e., odd)} \Rightarrow 2n+1 = 2(2k+1)+1 = 4k + \boxed{3}\right)$.

Thus, the alternating pattern really does continue!

C) If N and N' denote the two-digit numbers and a and b denote the digits, then $\begin{cases} N = 10a + b = 5k+1 \\ N' = 10b + a = 5j + 3 \end{cases}$. Adding, $N + N' = 11(a+b) = 5(j+k)+4$.

If $j+k$ is even, then $5(j+k)$ is a multiple of 10 and $a + b = 4$ or 14.

If $j+k$ is odd, $a + b = 9$. [19 is rejected, since the maximum digit sum is 9 + 9 = 18.]

$a+b=4 \Rightarrow N =$ ~~13~~, ~~22~~, 31

$a+b=9 \Rightarrow N =$ ~~18~~, ~~27~~, 36, ~~45~~, ~~54~~, ~~63~~, ~~72~~, 81

$a+b=14 \Rightarrow N =$ ~~77~~, 86, ~~95~~

The sum of all the numbers satisfying the specified conditions is 234 which leaves a remainder of **0** when divided by 9.

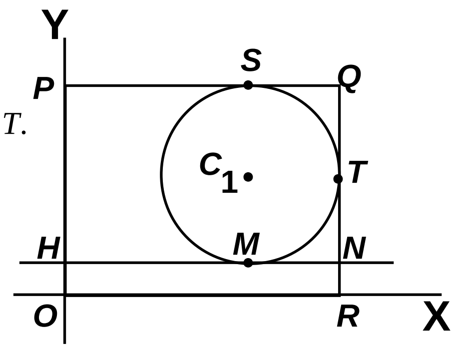
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2016
ROUND 3 COORDINATE GEOMETRY OF LINES AND CIRCLES

ANSWERS

A) _______________________________

B) (_____________ , _____________)

C) _______________________________



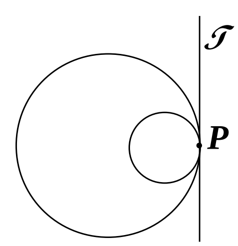
A) The equation of circle C_1 is $(x-6)^2+(y-4)^2=5.0625$.
Two sides of rectangle *OPQR* are tangent to circle C_1 at points *S* and *T*.
$\overleftrightarrow{HN} \parallel \overline{OR}$ and intersects circle C_1 at point *M*.
Compute the perimeter of rectangle *HORN*.

Recall: $\frac{1}{8}=0.125$.

B) Line $\mathcal{L}$ passes through points $A(-4,1)$ and $B(17,8)$. The line perpendicular to $\mathcal{L}$ has *x*-intercept at $C(3,0)$ and intersects $\mathcal{L}$ at $D(p,q)$. Compute the ordered pair (p,q).

C) Given: Circle $C_1: x^2+y^2=676$
How many unit circles, i.e. with radius 1, are internally tangent to C_1 and have a center at a lattice point?

Note: Two circles are internally tangent if they share a common tangent line $\mathcal{T}$ and the centers of the circles are on the same side of the tangent line $\mathcal{T}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2016 SOLUTION KEY

Round 3

A) Since the center of C_1 is at (6, 4) and $r^2 = 5\frac{1}{16} = \frac{81}{16}$, we have the radius is $\frac{9}{4} = 2.25$

Q(8.25, 6.25) and M(6,1.75).
Thus, rectangle *HORN* is 1.75 x 8.25, resulting in a perimeter of 2(1.75 + 8.25) = **<u>20</u>**.

B) $m_{\mathcal{L}} = \frac{8-1}{17+4} = \frac{1}{3}$.

The equation of $\mathcal{L}$ is $x - 3y = -7$.

The equation of the perpendicular is $(y-0) = -3(x-3)$

$\Rightarrow 3x + y = 9$ ***.

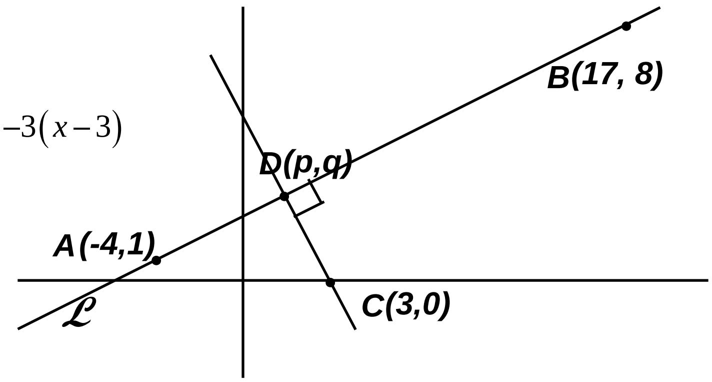
Solving simultaneously,

$3x + y = 9$
$x - 3y = -7$
$4x - 2y = 2$
$2x - y = 1$***

Adding,

$5x = 10 \Rightarrow x = 2, y = 3$

Thus, $(p, q) = \underline{(\mathbf{2,3})}$.

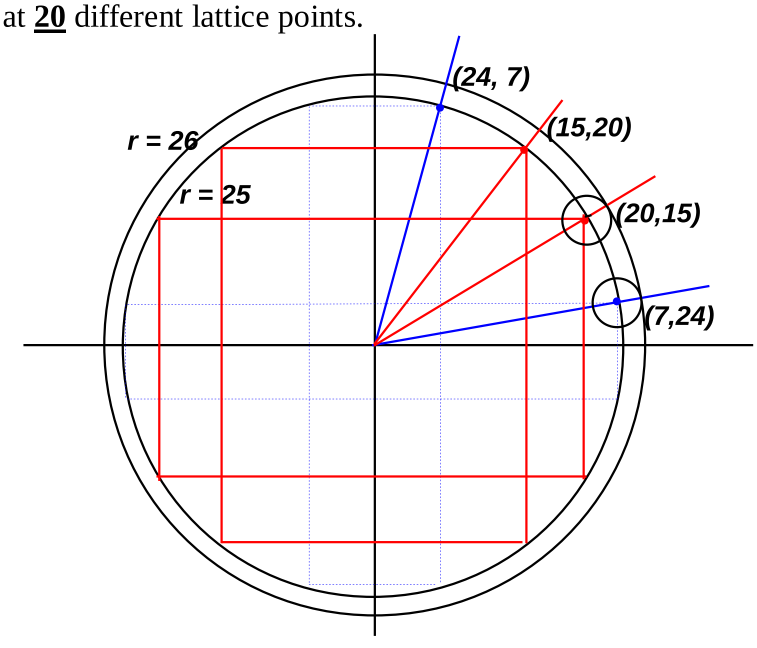


C) The given circle is origin-centered with radius 26. Thus, we are looking at two concentric origin-centered circles of radii 25 and 26. The equations of the required unit circles must be of the form $(x-h)^2 + (y-k)^2 = 1$, where $h^2 + k^2 = 25^2$.

This suggests two possible Pythagorean Triples: 7-24-25 and 5(3-4-5) = (15-20-25)
For each triple there are 8 possibilities: two in each of the 4 quadrants, as the coordinates of the center are swapped and the signs are changed from (+,+) to (-,+), (-,-) and (+,-).

We also must consider $(\pm 25, 0)$ and $(0, \pm 25)$.

Therefore, the center can be located at **<u>20</u>** different lattice points.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2016
ROUND 4 ALG 2: LOG & EXPONENTIAL FUNCTIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Compute the value(s) of x that satisfy the equation $\log_2 x + \log_2 \frac{1}{4} = \frac{3}{2}\log_2 25$.

B) If $\log_3\left(\log_2\left(\log_2 x\right)\right) = 1$, compute $\left(\log_4 x\right)^{\frac{1}{2}} \cdot \log_2 x$.

C) Determine the <u>domain</u> of the real-valued function defined by $y = \log_{10}\left(\frac{x^3+1}{x^3-x}\right)$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2016 SOLUTION KEY

Round 4

A) $\log_2 x + \log_2 \frac{1}{4} = \frac{3}{2}\log_2 25 \Leftrightarrow \log_2 \frac{x}{4} = \log_2\left(25^{\frac{3}{2}}\right) = \log_2 125 \Rightarrow \frac{x}{4} = 125 \Rightarrow x = \underline{\mathbf{500}}$.

B) $\log_3\left(\log_2\left(\log_2 x\right)\right) = 1 \Rightarrow \log_2\left(\log_2 x\right) = 3^1 \Rightarrow \log_2 x = 2^3 = 8 \Rightarrow x = 256$

$\left(\log_4 256\right)^{\frac{1}{2}} \cdot \log_2 256 = \sqrt{4} \cdot 8 = \underline{\mathbf{16}}$.

C) $x^3 - x = x(x+1)(x-1) \neq 0 \Rightarrow x \neq 0, \pm 1$

$$\frac{x^3+1}{x^3-x} = \frac{\cancel{(x+1)}\left(x^2-x+1\right)}{x\cancel{(x+1)}(x-1)} = \frac{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}{x(x-1)}$$

As a real-valued function, $y = \log_{10}\left(\frac{x^3+1}{x^3-x}\right)$ must have a positive argument.

$\Rightarrow \frac{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}{x(x-1)} > 0$. Since the numerator is always positive, the denominator determines the sign of the quotient. For $x < 0$ or $x > 1$, both factors in the denominator have the same parity (i.e. both are positive or both are negative) and the quotient will be positive.

Thus, the domain (*which must exclude* -1) is $\underline{\boldsymbol{x < -1,\ -1 < x < 0,\ x > 1}}$.

Interval notation is also acceptable:

$(-\infty, -1), (-1, 0), (1, \infty)$

Commas may be replaced by "or"s. Also accept $x < 0,\ x > 1\ (x \neq -1)$ or , since "and"s are evaluated before "or"s, $x < 0$ and $(x \neq -1)$ or $x > 1$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2016
ROUND 5 ALG 1: RATIO, PROPORTION OR VARIATION

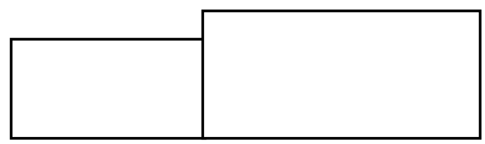
ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) Suppose the amount of light reflected by a set of mirrors is directly proportional to the surface area of the mirrors. 50 lumens of light are reflected off a mirror whose dimensions are 3 inches by 5 inches. How many lumens of light are reflected off two mirrors placed side-by-side made of the same material given that the first is 4 inches by 6 inches and the second is 5 inches by 9 inches?



B) Given: $\begin{cases} \dfrac{A}{B} = \dfrac{7}{9} \\ \dfrac{C}{D} = \dfrac{5}{3} \end{cases}$ If $B = 4D$, compute $\dfrac{A+B}{C+D}$.

C) As a 3-point shooter in basketball, I have currently hit on 60% of my attempts. If 8 of my hits had been misses, I would have only been a 50% 3-point shooter. Suppose I hit on my next k 3-point shots. What is the <u>minimum</u> value of k for which I hit at least 70% of my 3-point attempts?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2016 SOLUTION KEY

Round 5

A) $\frac{50}{3\cdot 5}=\frac{10}{3}$ lumens per in^2. $\frac{10}{3}\left(24+45\right)=80+150=\underline{\mathbf{230}}$ lumens.

B) $\frac{A+B}{C+D}=\frac{\frac{7}{9}B+B}{\frac{5}{3}D+D}=\frac{\frac{16}{9}B}{\frac{8}{3}D}=\frac{2B}{3D}=\frac{2\left(4D\right)}{3D}=\underline{\mathbf{\frac{8}{3}}}$.

C) Assume currently I have hit x 3-pointers in y attempts. Then:

$$\begin{cases}\frac{x}{y}=0.6=\frac{3}{5}\\ \frac{x-8}{y}=\frac{1}{2}\end{cases}\Leftrightarrow\begin{cases}5x=3y\\ y=2x-16\end{cases}\Leftrightarrow 5x=3\left(2x-16\right)\Rightarrow x=48,\ y=80$$

$$\frac{48+k}{80+k}\geq 0.70=\frac{7}{10}\Leftrightarrow 480+10k\geq 560+7k\Rightarrow 3k\geq 80\Rightarrow k_{\min}=\underline{\mathbf{27}}.$$

Check: $k=26\Rightarrow\frac{74}{106}\approx 0.6981,\ k=27\Rightarrow\frac{75}{107}\approx 0.7009$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2016
ROUND 6 PLANE GEOMETRY: POLYGONS (no areas)

ANSWERS

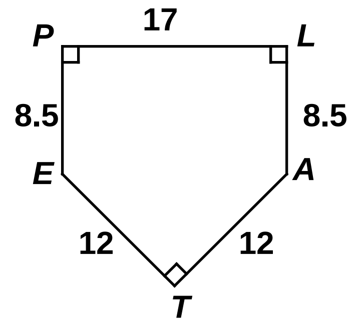
A) ______________________________

B) __________ A __________ O __________ R

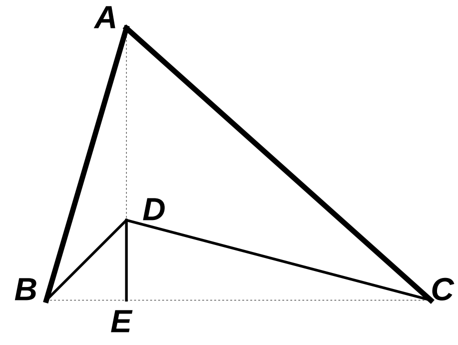
C) (______________ , ______________)

A) In a polygon with n sides, the ratio of the sum of the measures of the exterior angles (one at each vertex) to the sum of the measure of the interior angles is $\frac{1}{8}$. How many diagonals does this polygon have, *originating from any single vertex*?

B) In baseball, home plate according to the MLB rulebook, has 3 right angles and dimensions shown at the right. Rules may be rules, but, as students of mathematics, we realize that *this shape cannot exist*. $m\angle ETA$ may be close to 90°, but $\angle ETA$ is **not** a right angle. Is interior $\angle ETA$ Acute, Obtuse or Reflexive? Circle the correct letter in the answer blank above.



C) The diagonals of quadrilateral $ABCD$ (*segments* $\overline{AD}$ and $\overline{BC}$) do not intersect. This is always the case for a *concave* quadrilateral.
Note: Points A, B, C, D, and E all lie in the same plane.
Assume E lies on $\overline{BC}$.
If $\overline{ADE} \perp \overline{BC}$, $AB = BC = 25$, $AC = 30$, $BD = BE + 3$, $AE = 3(BE + 1)$, compute the ordered pair (DC, d), where d denotes the absolute value of the difference between the lengths of the diagonals.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2016 SOLUTION KEY

Round 6

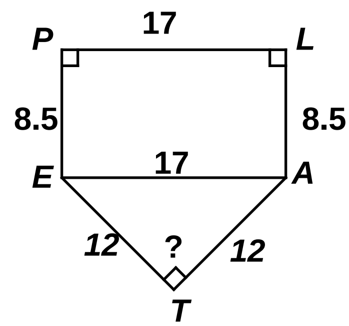
A) $\dfrac{360}{180(n-2)} = \dfrac{2}{n-2} = \dfrac{1}{8} \Rightarrow n-2=16 \Rightarrow n=18$.

Diagonals in a polygon <u>from</u> any single vertex can be drawn <u>to</u> any <u>other</u> vertex, except the two adjacent vertices, eliminating 3 vertices. Thus, 18 - 3 = <u>**15**</u>.

B) Reflexive refers to an angle whose measure is greater than 180° and less than 360°.
The <u>exterior</u> angle at *T* is a reflexive angle.
The interior angle must be either acute or obtuse.

If $\angle ETA$ *were* a right, then angle $EA^2 = \left(12\sqrt{2}\right)^2 = 144 \cdot 2 = 288$.

However, $EA^2 = 17^2 = 289$. Since the square of the actual length is slightly longer, $\angle ETA$ must be <u>**obtuse**</u>.



C) Either by invoking the Pythagorean theorem,

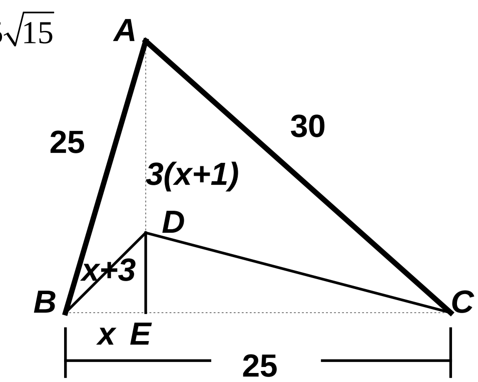
$x^2 + 9(x+1)^2 = 625 \Rightarrow 10x^2 + 18x - 616 = 0 \Rightarrow 5x^2 + 9x - 308 = (5x+44)(x-7) = 0 \Rightarrow x = 7$,

or, recognizing special right triangles (7-24-25, 18-24-30), we have $CE = 18, AE = 24$

$\Rightarrow BD = 10 \Rightarrow DE = \sqrt{51} \Rightarrow DC^2 = 51 + 18^2 = 375 \Rightarrow DC = 5\sqrt{15}$

$d = 25 - \left(24 - \sqrt{51}\right) = 1 + \sqrt{51}$

Thus, the required ordered pair is <u>$\left(\mathbf{5\sqrt{15}, 1+\sqrt{51}}\right)$</u>.

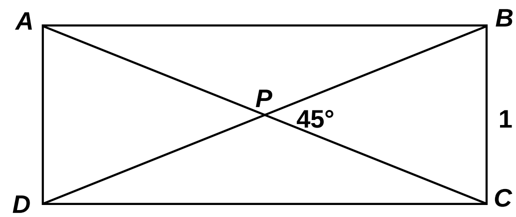


MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2016
ROUND 7 TEAM QUESTIONS
ANSWERS

A) (______ , ______ , ______) D) ____________________________

B) ____________________________ E) (___________ , ___________)

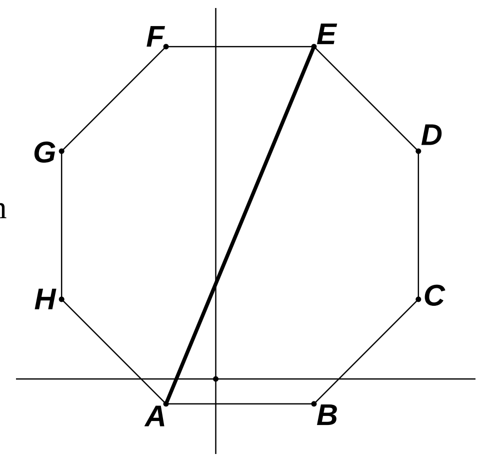
C) (___________ , ___________) F) (_______ , _______ , _______)



A) The perimeter of the rectangle $ABCD$, where $BC = 1$, and the diagonals intersect at a 45° angle may be expressed in the form $s + t\sqrt{r}$, where $\sqrt{r}$ is a simplified radical. Compute the ordered triple (s,t,r).

B) Find <u>all</u> integer ordered pairs (n, k) which satisfy the following equation:

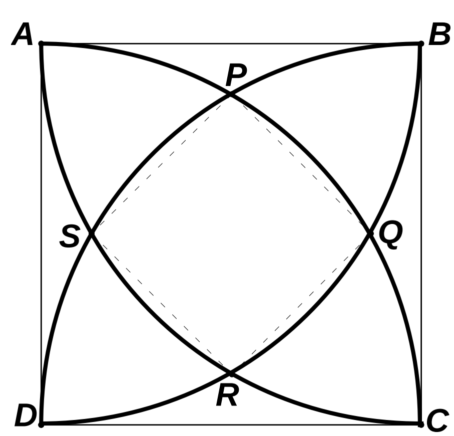
$$n^3 - 3n^2 + 3n = k^3 + 3k^2 + 3k - 17$$



C) A <u>regular</u> octagon is placed on a coordinate system as shown in the diagram. If $A(-2,-1)$ and $B(4,-1)$, compute the coordinates of the x-intercept of $\overline{AE}$.

D) Given: $\log 250 = N$
Give a <u>simplified</u> expression for $\log_5 250$ in terms of N.

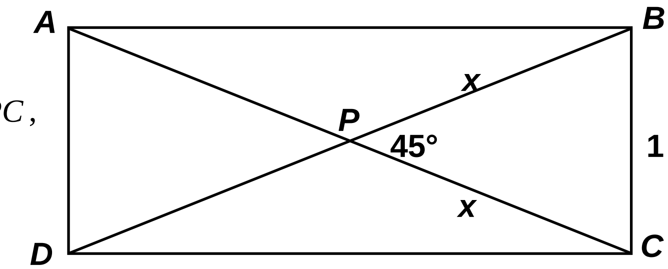
E) The golden rectangle is a rectangle with dimensions L_g x W_g, where $L_g > W_g$ which satisfies the proportion $\frac{L_g}{W_g} = \frac{L_g + W_g}{L_g}$. Define a silver rectangle to be a rectangle with dimensions L_s x W_s, where $L_s > W_s$ and diagonal D_s which satisfies the proportion $\frac{L_s}{W_s} = \frac{D_s}{L_s}$. The ratio $\frac{L_g}{W_g} : \frac{L_s}{W_s} = \frac{\sqrt{B}}{A}$, where $A > 1$ is an integer and $A \cdot B$ is a minimum. Compute the ordered pair (A,B).



F) Given square $ABCD$ with side of length 4.
Arcs are drawn from each vertex and the points of intersection form square $PQRS$ as shown. In simplified radical form, the perimeter of $PQRS$ is $a\left(\sqrt{b} - \sqrt{c}\right)$, where a, b, and c are positive integers.
Compute the ordered triple (a,b,c).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2016 SOLUTION KEY

Team Round



A) Solution #1:

Let $PB = PC = x$. Using the Law of Cosines on ΔBPC,

$$x^2 + x^2 - 2x^2\cos 45° = 1 \Rightarrow \left(2-\sqrt{2}\right)x^2 = 1$$

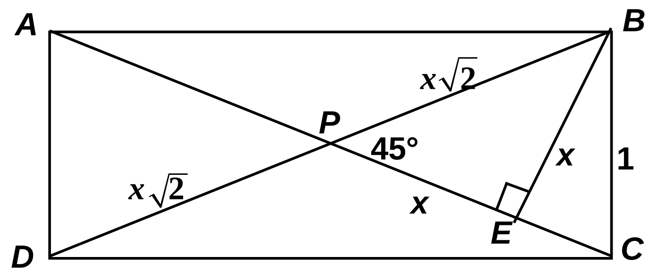
$$\Rightarrow x^2 = \frac{1}{2-\sqrt{2}} = \frac{2+\sqrt{2}}{2}$$

$$\Rightarrow BD = 2x = 2\sqrt{\frac{2+\sqrt{2}}{2}}$$

$$DC^2 = BD^2 - 1 = 4\left(\frac{2+\sqrt{2}}{2}\right) - 1 = 3 + 2\sqrt{2} = \left(1+\sqrt{2}\right)^2$$

$\Rightarrow$ Perimeter = $2\left(1+\sqrt{2}\right) + 2 = \mathbf{4 + 2\sqrt{2}} \Rightarrow (s,t,r) = \underline{\mathbf{(4,2,2)}}$.

Solution #2:



Drop a perpendicular from B to $\overline{AC}$

$EC = x\sqrt{2} - x = x\left(\sqrt{2}-1\right)$

In ΔBEC,

$$x^2 + \left(\sqrt{2}-1\right)^2 x^2 = 1^2 \Rightarrow x^2\left(1+2+1-2\sqrt{2}\right) = 1$$

$$\Rightarrow x^2 = \frac{1}{4-2\sqrt{2}} = \frac{4+2\sqrt{2}}{8} = \frac{2+\sqrt{2}}{4}$$

$$BD = 2x\sqrt{2} \Rightarrow BD^2 = 8x^2 = 8\left(\frac{2+\sqrt{2}}{4}\right) = 4 + 2\sqrt{2}$$

But $DC^2 = BD^2 - 1 = 4 + 2\sqrt{2} - 1 = 3 + 2\sqrt{2}$ and the same result follows.

Note that the perimeter is numerically equal to BD^2 !!

Solution #3

Note that $m\angle BDC = \frac{1}{2} m\angle BPC$. So, using $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$, we have

$$\tan\left(\angle BDC\right) = \frac{\sin 45°}{1+\cos 45°} = \frac{\sqrt{2}/2}{1+\sqrt{2}/2} = \frac{1}{1+\sqrt{2}} \Rightarrow DC = 1+\sqrt{2}$$ and the same result follows.

Team Round - continued

B) $n^3-3n^2+3n=k^3+3k^2+3k-17 \Leftrightarrow \left(n^3-3n^2+3n-1\right)=\left(k^3+3k^2+3k+1\right)-19$

$\Leftrightarrow (n-1)^3-(k+1)^3=-19$

As the difference of perfect squares, the left hand side of this equation factors as

$(n-k-2)\left[(n-1)^2+(n-1)(k+1)+(k+1)^2\right]=-19$

Since n and k are integers and 19 is prime, we have two possibilities,

(1) $\begin{cases} n-k-2=-19 \\ (n-1)^2+(k+1)^2+(n-1)(k+1)=1 \end{cases}$ or (2) $\begin{cases} n-k-2=-1 \\ (n-1)^2+(k+1)^2+(n-1)(k+1)=19 \end{cases}$

Notice we are tacitly assuming that the second equation must be equal to a positive number. The reasoning will be given at the end.

(1) $\Rightarrow n=-17+k$

Substituting, we have $(k-18)^2+(k+1)^2+(k-18)(k+1)=1 \Leftrightarrow 3k^2-51k+18(17)+1=0$

which has no rational solutions, since the discriminant is negative (-1083).

(2) $\Rightarrow n=k+1$

Substituting, we have

$k^2+(k+1)^2+k(k+1)=19 \Leftrightarrow 3k^2+3k-18=3\left(k^2+k-6\right)=3(k-2)(k+3)=0 \Rightarrow k=2,-3$

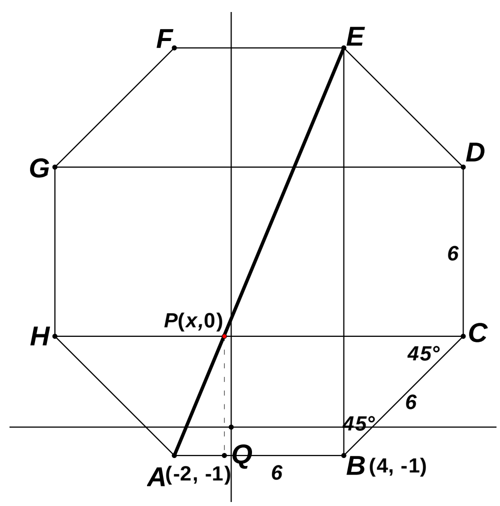
$\Rightarrow (n, k)=$ **(3, 2) or (–2, –3)**.

Clearly, an expression of the form A^2+AB+B^2 is positive if A and B are either both positive or both negative, but what about when they have opposite signs?

Completing the square, $A^2+AB+B^2=A^2+2AB+B^2-AB=(A+B)^2-AB$.

If A and B have opposite signs, then we are subtracting a negative and again the expression is positive.

C) Solution #1: $\Delta AQP \sim \Delta ABE$



$\dfrac{AQ}{AB}=\dfrac{PQ}{EB} \Leftrightarrow \dfrac{x+2}{6}=\dfrac{1}{6+2\left(3\sqrt{2}\right)}$

$\Rightarrow x+2=\dfrac{6}{6+6\sqrt{2}}=\dfrac{1}{1+\sqrt{2}}\cdot\dfrac{1-\sqrt{2}}{1-\sqrt{2}}=\sqrt{2}-1$

$\Rightarrow x=\sqrt{2}-3$

Thus, the x-intercept of $\overline{AE}$ is at $\mathbf{\left(-3+\sqrt{2},0\right)}$.

Solution #2:

The equation of $\overleftrightarrow{AE}$ is

$(y+1)=\dfrac{6+6\sqrt{2}}{6}(x+2)=\left(\sqrt{2}+1\right)(x+2)$. Substituting $y=0$ into $(y+1)=\left(\sqrt{2}+1\right)(x+2)$ gives the same result.

Team Round - continued

D) Let A denote $\log 2$.

$$\log_5 250 = \frac{\log 250}{\log 5} = \frac{N}{\log\left(\frac{10}{2}\right)} = \frac{N}{\log 10 - \log 2} = \underline{\mathbf{\frac{N}{1-A}}}$$

But $N = \log 250 = \log\left(2\cdot 5^3\right) = \log 2 + 3\log\left(\frac{10}{2}\right) = 3 - 2\log 2 = 3 - 2A \Rightarrow A = \frac{3-N}{2}$

Substituting, $\log_5 250 = \frac{N}{1-A} = \frac{N}{1-\frac{3-N}{2}} = \frac{N}{\frac{N-1}{2}} = \underline{\mathbf{\frac{2N}{N-1}}}$.

E) $\frac{L_g}{W_g} = \frac{L_g + W_g}{L_g} \Rightarrow L_g^{\,2} - L_g W_g - W_g^{\,2} = 0$. Dividing by $W_g^{\,2}$, $\left(\frac{L_g}{W_g}\right)^2 - \frac{L_g}{W_g} - 1 = 0$.

Applying the Q.F., $\frac{L_g}{W_g} = \frac{1+\sqrt{5}}{2} = n$.

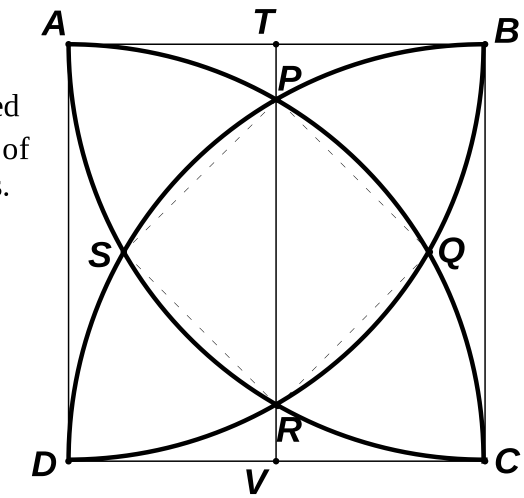
$$\frac{L_s}{W_s} = \frac{D_s}{L_s} \Rightarrow L_s^{\,2} = W_s\sqrt{L_s^{\,2} + W_s^{\,2}} \Rightarrow L_s^{\,4} = W_s^{\,2}\left(L_s^{\,2} + W_s^{\,2}\right)$$

Multiplying out, dividing by $W_s^{\,4}$ and transposing terms $\Rightarrow \left(\frac{L_s}{W_s}\right)^4 - \left(\frac{L_s}{W_s}\right)^2 - 1 = 0$. Applying the Q.F. again, we have $\left(\frac{L_s}{W_s}\right)^2 = \frac{1+\sqrt{5}}{2} = n$. Therefore, the required ratio is

$$\frac{n}{\sqrt{n}} = \sqrt{n} = \sqrt{\frac{1+\sqrt{5}}{2}\cdot\frac{2}{2}} = \sqrt{\frac{2+2\sqrt{5}}{4}} = \frac{\sqrt{2+2\sqrt{5}}}{2}$$

Multipliers other than $\frac{2}{2}$ were possible. $\frac{8}{8} \Rightarrow \frac{\sqrt{8+8\sqrt{5}}}{4}$, $\frac{18}{18} \Rightarrow \frac{\sqrt{18+18\sqrt{5}}}{6}$, but in all these cases, the product $A\cdot B$ is larger. Since it was given that $A > 1$, $A = 2$ is a minimum and we have $(A,B) = \underline{\mathbf{\left(2,\ 2+2\sqrt{5}\right)}}$.

Team Round - continued



F) Let D be the origin, $\overline{DC}$ lie on the positive x-axis, and $\overline{DA}$ lie on the positive y-axis.
R and S (and P and Q) are images across $y = x$, so we need only find the coordinates of one point and the coordinates of the other is found by simply interchanging the coordinates.
Clearly, the x-coordinate of points P and R is 2 and the y-coordinate of points Q and S is 2 as well.
The equation of the arc $\overset{\frown}{APQC}$ is $x^2 + y^2 = 16$.

$x = 2 \Rightarrow y = 2\sqrt{3}$

Therefore, the coordinates are $P\left(2, 2\sqrt{3}\right)$, $Q\left(2\sqrt{3}, 2\right)$.

$TP + PV = TV \Leftrightarrow TP + 2\sqrt{3} = 4 \Rightarrow TP = 4 - 2\sqrt{3}$

$TP = RV \Rightarrow R\left(2, 4-2\sqrt{3}\right)$, $S\left(4-2\sqrt{3}, 2\right)$

$PR = 2\sqrt{3} - \left(4 - 2\sqrt{3}\right) = 4\sqrt{3} - 4$

$$PQ = \frac{PR}{\sqrt{2}} = \frac{4\left(\sqrt{3}-1\right)}{\sqrt{2}} = 2\sqrt{2}\left(\sqrt{3}-1\right)$$

$\Rightarrow$ Perimeter $= 4PQ = 8\sqrt{2}\left(\sqrt{3}-1\right) = 8\left(\sqrt{6}-\sqrt{2}\right) \Rightarrow (a,b,c) = \underline{\mathbf{(8,\ 6,\ 2)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2016 ANSWERS

Round 1 Trig: Right Triangles, Laws of Sine and Cosine

A) 504 B) $\frac{4}{3}$ C) $\sqrt{7}$

Round 2 Arithmetic/Elementary Number Theory

A) (9, 27) B) 3 C) 0

Round 3 Coordinate Geometry of Lines and Circles

A) 20 B) $(2,3)$ C) 20

Round 4 Alg 2: Log and Exponential Functions

A) 500 B) 16 C) $x<-1,\ -1<x<0,\ x>1$

Round 5 Alg 1: Ratio, Proportion or Variation

A) 230 B) $\frac{8}{3}$ C) 27

Round 6 Plane Geometry: Polygons (no areas)

A) 15 B) obtuse C) $\left(5\sqrt{15},\ 1+\sqrt{51}\right)$

Team Round

A) $(4,2,2)$ D) $\frac{2N}{N-1}$

B) $(-2, -3)$ and $(3, 2)$ E) $\left(2,\ 2+2\sqrt{5}\right)$

C) $\left(-3+\sqrt{2},0\right)$ F) $(8,6,2)$