MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2017
ROUND 1 ANALYTIC GEOMETRY: ANYTHING

ANSWERS

A) ________________________________

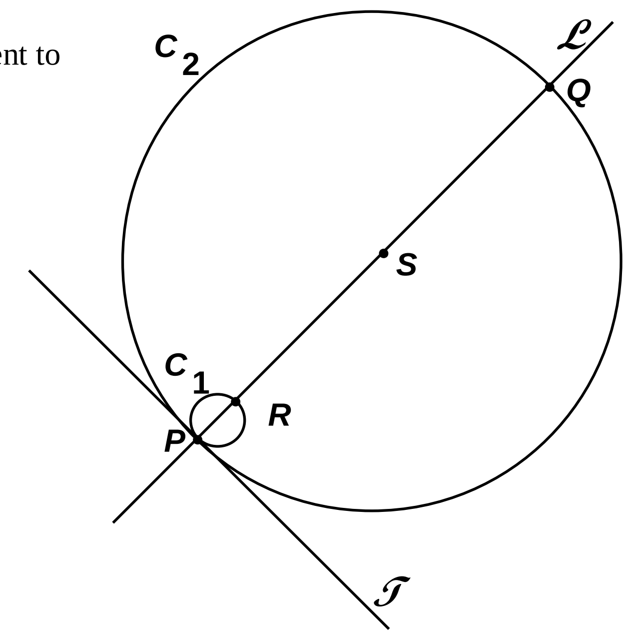
B) ________________________________

C) (_______ , ________ , _________)

A) The vertical line $x = 4$ and the horizontal line $y = -5$ intersect the hyperbola $xy = 60$ in points *P* and *Q*, respectively. $\overleftrightarrow{PQ}$ intersects the *x*-axis at $(h, 0)$. Compute *h*.

B) Let *B* and *V* denote the *y*-intercept and the vertex of the parabola $y = (x-4)^2 + k$. Compute *BV*.

C) Circle C_1 whose center is at $(2, -6)$ is internally tangent to circle C_2 at point $P(-2, -9)$.
$\mathscr{T}$ is the common tangent line.
Points *P*, *Q*, *R*, and *S* lie on $\mathscr{L}$.
$\mathscr{L} \perp \mathscr{T}$.
S is the center of C_2 whose radius is 40
The equation of the circle C_3 with center on $\mathscr{L}$, passing through point *R* and *S* has equation $(x-h)^2 + (y-k)^2 = r^2$.
Compute the ordered triple (h, k, r^2).



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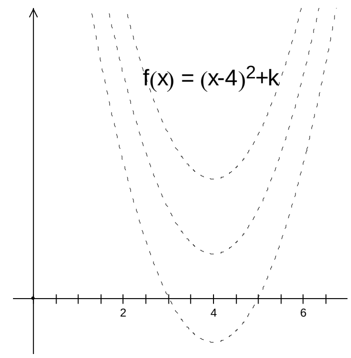
CONTEST 4 - JANUARY 2017 SOLUTION KEY

Round 1

A) $P(4,15)$, $Q(-12,-5)$. The slope of the line is $m=\dfrac{15-(-5)}{4-(-12)}=\dfrac{20}{16}=\dfrac{5}{4}$.

Therefore, a change of +4 in *x* produces a change of +5 in *y*.
Starting at *Q*, we have $(-12+4,-5+5)=(-8,0)\Rightarrow h=\underline{\mathbf{-8}}$.



B) Given: $y=(x-4)^2+k$

By inspection, the vertex is at $V(4,k)$. By letting $x=0$, $B(0,16+k)$.

Therefore, $$BV=\sqrt{(4-0)^2+(k-(16+k))^2}=\sqrt{16+(-16)^2}=\sqrt{2^4+2^8}$$ $$=\sqrt{2^4(1+2^4)}=\underline{\mathbf{4\sqrt{17}}}.$$

C) Since the center of C_1 must lie on $\mathcal{L}$,

$m=\dfrac{3}{4}$, $\Delta x=+4$, $\Delta y=+3$.

$P(-2,-9)\rightarrow C_1(2,-6)\rightarrow R(2+4,-6+3)=\boxed{R(6,-3)}$

$r_2=40\Rightarrow PQ=80=16\cdot 5$

Using a slope of $\dfrac{3}{4}$ and *similar* triangles,

$16(4\text{-}3\text{-}5)\Rightarrow 64\text{-}48\text{-}80\ \Rightarrow$

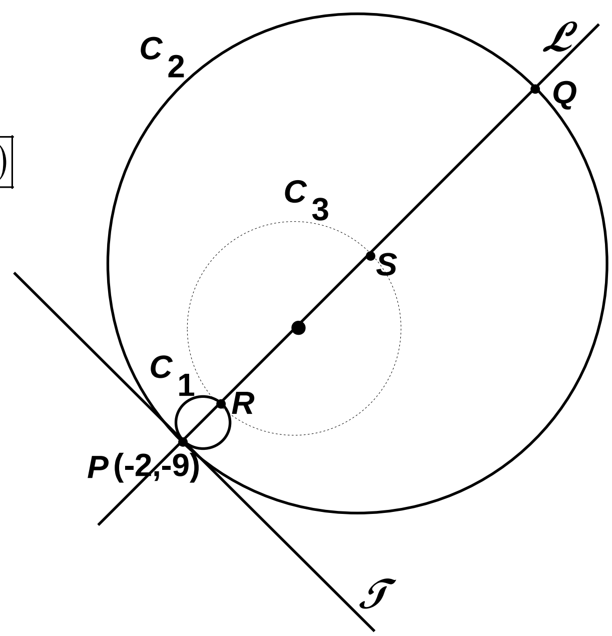
$(-2+64,-9+48)=Q(62,39)$

S is the midpoint of $\overline{PQ}\ \Rightarrow$

$\left(\dfrac{-2+62}{2},\dfrac{-9+39}{2}\right)=\boxed{S(30,15)}$

The center of C_3 is the midpoint of $\overline{RS}\ \Rightarrow$

$\left(\dfrac{6+30}{2},\dfrac{-3+15}{2}\right)=C_3(18,6)$



The radius is the distance from $(18,6)$ to $(30,15)$ which is $\sqrt{12^2+9^2}=\sqrt{225}=15$.

Thus, the required ordered triple is $\underline{\mathbf{(18,6,225)}}$.

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ROUND 2 ALG 1: FACTORING AND/OR EQUATIONS INVOLVING FACTORING

ANSWERS

A) _______________________________

B) _______________________________

C) _______________________________

A) For positive integer values of *k*, neither $10+k$ nor $10-k$ is prime, but they are positive and do have the same number of divisors. Determine all possible values of *k*.

B) Given: $A + B = 7$ and $A^2B - 21 = -AB^2$
Compute $A^2 - AB + B^2$.

C) Find all real values of *x* for which the harmonic mean of $2x+1$ and $\dfrac{1}{x-1}$ is $\dfrac{6}{5}$.

Recall: The harmonic mean of *a* and *b* is defined as $\dfrac{2ab}{a+b}$.

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Round 2

A) For $k = 1, 2, 3, \ldots, 9$, $(10+k, 10-k) = (11,9), (12,8), (13,7), (14,6), \ldots (19,1)$, where each pair of coordinates adds up to 20. Since both 14 and 6 have exactly 4 divisors, $k = 4$ works. It is left to you to verify that all other pairs contain one or more prime numbers, or the numbers have a different number of divisors, so $k = \underline{\mathbf{4}}$ is the only solution.

B) Since we were given $A + B = 7$, $A^2B + AB^2 = AB(A+B) = 21 \Rightarrow AB = 3$.
To evaluate $A^2 - AB + B^2$, we notice that, since $(A+B)^2 = A^2 + 2AB + B^2$, we can subtract $3AB$ to get the required expression. Thus, $A^2 - AB + B^2 = (A+B)^2 - 3AB = 7^2 - 3 \cdot 3 = \underline{\mathbf{40}}$.

Solution #2:
$A + B = 7 \Rightarrow (A+B)^3 = A^3 + B^3 + 3(A^2B + AB^2) = 343$.
Thus, $A^3 + B^3 + 3(21) = 343 \Rightarrow A^3 + B^3 = 280$. $A^2 - AB + B^2 = \dfrac{A^3 + B^3}{A+B} = \dfrac{280}{7} = \underline{\mathbf{40}}$.

C) $$\frac{2 \cdot \frac{2x+1}{x-1}}{2x+1+\frac{1}{x-1}} = \frac{6}{5} \Rightarrow 10\left(\frac{2x+1}{x-1}\right) = 6(2x+1) + \frac{6}{x-1} \Rightarrow 20x + 10 = 6(2x+1)(x-1) + 6$$
$$12x^2 - 26x - 10 = 2\left(6x^2 - 13x - 5\right) = 2(3x+1)(2x-5) = 0 \Rightarrow x = \underline{-\frac{\mathbf{1}}{\mathbf{3}}, \frac{\mathbf{5}}{\mathbf{2}}}.$$

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ROUND 3 TRIG: EQUATIONS WITH A REASONABLE NUMBER OF SOLUTIONS

ANSWERS

A) ______________________________ °

B) ______________________________

C) ______________________________

A) Given: $\cos 36° = \dfrac{\sqrt{5}+1}{4}$

Solve for x, where $180° < x < 270°$: $\sin(x+6°) = -\dfrac{\sqrt{5}+1}{4}$

B) Solve over $0 < x < \pi$: $4\sin^3 x - 4\sin^2 x - 3\sin x + 3 = 0$

C) Given: $\left(4^{\sin^2 x}\right)\left(4^{\cos^2 x}\right)\left(4^{\tan^2 x}\right) = \sqrt[3]{128}$

Compute $\cos 2x$

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Round 3

A) $\sin(x+6°) = -\cos(36°) \Leftrightarrow \sin(x+6°) = -\sin(54°) = \sin(180+54) = \sin(234°)$

$x + 6° = 234° \Rightarrow x = \underline{\mathbf{228}}°$.

B) $4\sin^3 x - 4\sin^2 x - 3\sin x + 3 = 4\sin^2 x\left(\sin x - 1\right) - 3\left(\sin x - 1\right) = \left(\sin x - 1\right)\left(4\sin^2 x - 3 = 0\right)$

$\Rightarrow \sin x = 1, \pm\frac{\sqrt{3}}{2} \Rightarrow x = \underline{\boldsymbol{\frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}}}$. [$\sin x = -\frac{\sqrt{3}}{2}$ is impossible over the stated domain.]

C) $\left(4^{\sin^2 x}\right)\left(4^{\cos^2 x}\right)\left(4^{\tan^2 x}\right) = 4^{\sin^2 x + \cos^2 x + \tan^2 x} = 4^{1+\tan^2 x} = 4^{\sec^2 x} = 2^{2\sec^2 x} = \sqrt[3]{128} = \sqrt[3]{2^7} = 2^{\frac{7}{3}}$

$\Rightarrow \sec^2 x = \frac{7}{6} \Rightarrow \cos^2 x = \frac{6}{7}$.

Since $\cos 2x = 2\cos^2 x - 1$, we have $2\left(\frac{6}{7}\right) - 1 = \frac{12}{7} - 1 = \underline{\boldsymbol{\frac{5}{7}}}$.

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ROUND 4 ALG 2: QUADRATIC EQUATIONS

ANSWERS

A) (_______ , ________ , _________)

B) ______________________________

C) (_______ , ________ , _________)

A) One solution of the quadratic equation $Ax^2 + Bx + C = 0$ is $\dfrac{5 + i\sqrt{3}}{2}$.
If A, B and C are integers, $A > 0$, and the greatest common factor of A, B and C is 1, compute the ordered triple (A, B, C).

B) J and K are positive integers.
The ordered pair $(x, y) = (J, K)$, where $J + K < 100$, satisfies the equation $x + 2 = y^2 - 7$.
Compute the sum of the largest and smallest possible values of J.

C) The quadratic equation $Ax^2 + Bx + C = 0$, where A, B and C are relatively prime integers and $A > 0$, has roots $r_1 = \sqrt{\dfrac{40}{9} + \dfrac{41}{4}}$ and $r_2 = B + 2$.
Compute the ordered triple (A, B, C).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2017 SOLUTION KEY

Round 4

A) With integer coefficients, the roots must be conjugates of each other, i.e. the roots are $\frac{5 \pm i\sqrt{3}}{2}$ and the sum of the roots is 5 and the product of the roots is $\frac{25 - i^2 \cdot 3}{4} = \frac{25+3}{4} = 7$
Therefore, the equation is $x^2 - 5x + 7 = 0$ and $(A, B, C) = \underline{\mathbf{(1,-5,7)}}$.

B) $x + 2 = y^2 - 7 \Leftrightarrow y^2 = x + 9$
$x = 7$ is the smallest positive value of x for which y is also an integer.
The first few (J, K) are: (<u>7</u>, 4), (16, 5), (27, 6), (40, 7), …
Note that, as the y-values increase by 1, the <u>gap</u> between the x-values increases by 2.
Therefore, subsequent pairs are: (40+15, 7+1) = (55,8), (55+17,8+1) = (<u>72</u>, 9), (72+19, 9+1) = (91, 10), …
We stop here, since $J + K > 100$.
The required sum is 7 + 72 = <u>**79**</u>.

C) $r_1 = \sqrt{\frac{40}{9} + \frac{41}{4}} = \sqrt{\frac{4(40)+9(41)}{4 \cdot 9}} = \sqrt{\frac{160+369}{36}} = \sqrt{\frac{529}{36}} = \sqrt{\frac{23^2}{6^2}} = \frac{23}{6}$
Thus, one factor is $6x - 23$. Let's assume the other factor is $x - r_2$.
$(6x-23)(x-r_2) = 6x^2 - (23+6r_2)x + 23r_2$ and $B = -23 - 6r_2 = r_2 - 2 \Rightarrow r_2 = -3$
$\Rightarrow (A, B, C) = \underline{\mathbf{(6,-5,-69)}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2017
ROUND 5 GEOMETRY: SIMILARITY OF POLYGONS

ANSWERS

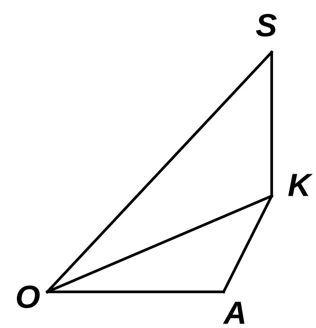
A) _______________________________

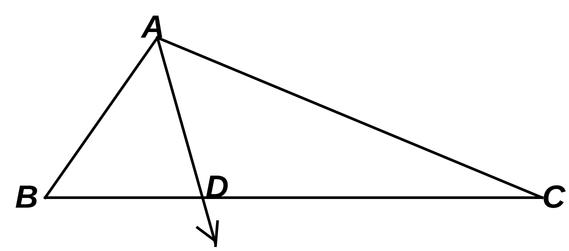
B) _______________________________

C) _______________________________

Diagrams are not necessarily drawn to scale.

A) Given: $\angle SOK \cong \angle KOA$, $OS = 9$, $OK = 6$, $OA = 4$ and the perimeter of quadrilateral $SOAK$ is 21.
Compute KS.





B) In ΔABC, $AB = x - 2$, $AC = 2x$, and $BC = 2x + 1$
Point D is on $\overline{BC}$ such that $\overrightarrow{AD}$ bisects $\angle BAC$.
Equilateral triangles are constructed on $\overline{BD}$ and $\overline{DC}$.
If the perimeter of ΔABC is 49, the <u>simplified</u> ratio of the area of the larger equilateral triangle to the area of the smaller equilateral triangle is $K : J$. Compute $K + J$.

C) Triangle T has an area of 1 square unit, and it is similar to a right triangle with sides of length 3, 4, and 5. T is inscribed in a circle. Compute the area of this circle.

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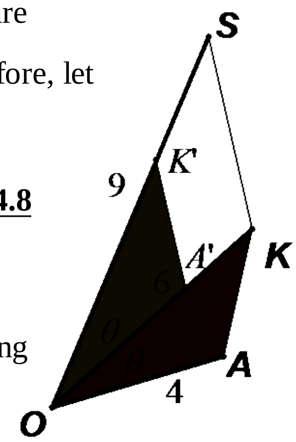
Round 5

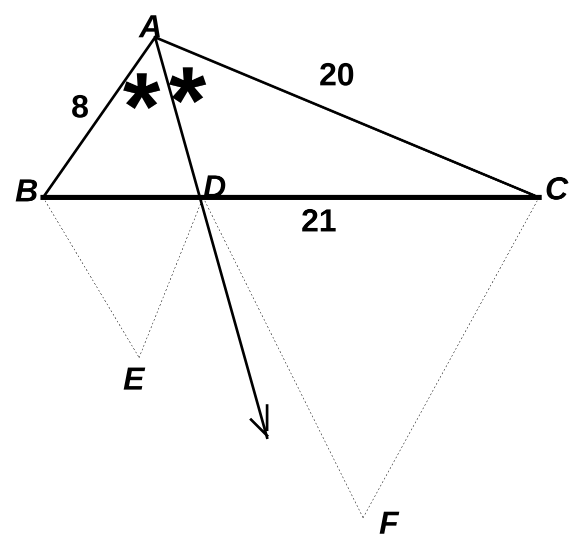


A) Since $\angle SOK \cong \angle KOA$ and the lengths of the sides that include these angles are proportional, namely, $\left(\frac{6}{4}=\frac{9}{6}\right)$, $\Delta SOK \sim \Delta KOA$ by SAS and $\frac{KS}{AK}=\frac{3}{2}$ Therefore, let $KA = 2x$ and $KS = 3x$.

The perimeter of *SOAK* is $4+9+5x = 21 \Rightarrow 5x = 8 \Rightarrow x = \frac{8}{5} \Rightarrow KS = \underline{\boldsymbol{\frac{24}{5}}}$ or **<u>4.8</u>**

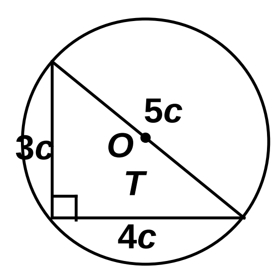
Why does SAS~ work?
Pivot (rotate) ΔKOA about point O through an angle θ and $\overline{A'K'} \parallel \overline{SK}$, resulting the more common scenario where similar triangles are formed when a line passes through two sides of a triangle parallel to the third side.

B) $(x-2)+(2x)+(2x+1) = 49 \Rightarrow x = 10$
$\Rightarrow (AB, AC, BC) = (8, 20, 21)$
According to the angle bisector theorem,
$\frac{BD}{DC} = \frac{AB}{AC} = \frac{8}{20} = \frac{2}{5}$.
Without determining the lengths of $\overline{BD}$ and $\overline{DC}$, we know the required ratio is 4 : 25 which gives us $K + J = \underline{\mathbf{29}}$.



C) Since the 3-4-5 triangle has an area of 6, the sides of T are scaled by a factor of $\frac{1}{\sqrt{6}}$. Inscribing T in a circle means that the hypotenuse of T will be the diameter of the circle.

Thus, the area of the circle is $\pi r^2 = \pi\left(\frac{1}{2}\cdot\frac{5}{\sqrt{6}}\right)^2 = \underline{\boldsymbol{\frac{25\pi}{24}}}$.



MASSACHUSETTS MATHEMATICS LEAGUE
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ROUND 6 ALG 1: ANYTHING

ANSWERS

A) ______________________

B) ______________________ (base 11)

C) ______________________

A) Compute the non-integer solution of $\frac{1}{T} = \frac{2T-1}{15}$.

B) If $3KK_{(\text{base } 8)} = 4KJ_{(\text{base } 7)}$, compute the sum of all possible values of *K*. Write your answer in base 11.

C) Both *a* and *b*, where $a < b$, have the property that twice each number, diminished by its reciprocal, is $\frac{1}{6}$. Let *N* denote the largest of $a+b$, $a-b$, ab, and $\frac{a}{b}$. Compute *N*.

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Round 6

A) Cross multiplying, $\frac{1}{T}=\frac{2T-1}{15} \Leftrightarrow 2T^2-T=15$, provided $T\neq 0$.

$2T^2-T-15=(2T+5)(T-3)=0 \Rightarrow T=\underline{\mathbf{-\frac{5}{2}}}$, $\cancel{3}$ (non-integer values only)

B) $3KK_{(8)}=3(8^2)+K(8^1)+K(8^0)=192+9K$

$4KJ_{(7)}=4(7^2)+K(7^1)+J(7^0)=7K+J+196$

Equating, $2K-J=4$

Since J must be a digit in base 7, $0\leq J\leq 6$

Since K must be a digit in both base 7 and base 8, we also have $0\leq K\leq 6$.

Thus, $(K,J)=(2,0),(3,2),(4,4),(5,6)$

$2+3+4+5=14_{(10)}=\underline{\mathbf{13}}_{(11)}$

C) $2x-\frac{1}{x}=\frac{1}{6} \Rightarrow \frac{2x^2-1}{x}=\frac{1}{6}$

Cross multiplying, $12x^2-6=x \Leftrightarrow 12x^2-x-6=(4x-3)(3x+2)=0$

$x=\frac{3}{4},-\frac{2}{3}$ and $a<b \Rightarrow (a,b)=\left(-\frac{2}{3},\frac{3}{4}\right)$.

Since $a-b<0$, $ab<0$, and $\frac{a}{b}<0$, the winner is $a+b=\frac{3}{4}-\frac{2}{3}=\frac{9-8}{12}=\underline{\mathbf{\frac{1}{12}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
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ROUND 7 TEAM QUESTIONS

ANSWERS

A) ________________________________ D) S = ______ (x, y) = (____ , ____)

B) ____ E) ____________________

C) ________________________________ F) ____________________________

A) Compute the ***y*-coordinates** of all points where the vertical line $x=-7$ could intersect a hyperbola which has an asymptote whose equation is $(y-4)=\frac{3}{4}(x+2)$.

B) Factor completely over the integers. $4x^6-13x^4-13x^2+4$

C) Given: $\sin(x)=\frac{1\pm\sqrt{1-4(N-1)}}{2}$
If $x = (30k)°$, where k is an integer and $0 \le k < 12$.
Compute all possible rational values of N.

D) Given: $S=x^2+4xy+9y^2+4x+18y+2017$ for real numbers x and y.
Find m, the minimum value of S, and the ordered pair (x, y) for which it occurs.

E) In right triangle ABC, the bisector of $\angle C$ intersects the hypotenuse $\overline{AB}$ at point D.
Point E is located on $\overline{AC}$ such that $\overline{DE}\perp\overline{AC}$. Compute $\frac{1}{BC}+\frac{1}{AC}$, if $DE=\frac{\sqrt{5}+1}{2}$.

F) Some positive factors of 4000 are multiples of 5 and some are not.
Specifically, k% of the factors of 4000 are multiples of 5.
A is a positive integer less than 4000 with the same number of divisors as 4000, and k% of its divisors are multiples of 5.
B is an integer greater than 4000 with the same number of divisors as 4000, and k% of its divisors are multiples of 5.
Compute the minimum possible value of $B – A$.

MASSACHUSETTS MATHEMATICS LEAGUE
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Team Round

A) The center of the hyperbola is $(-2,4)$, but the hyperbola could be vertical or horizontal.

Case 1: Vertical $(a,b)=(3,4)$

$$\frac{(y-4)^2}{9}-\frac{(x+2)^2}{16}=1$$

For $x=-7$, we have

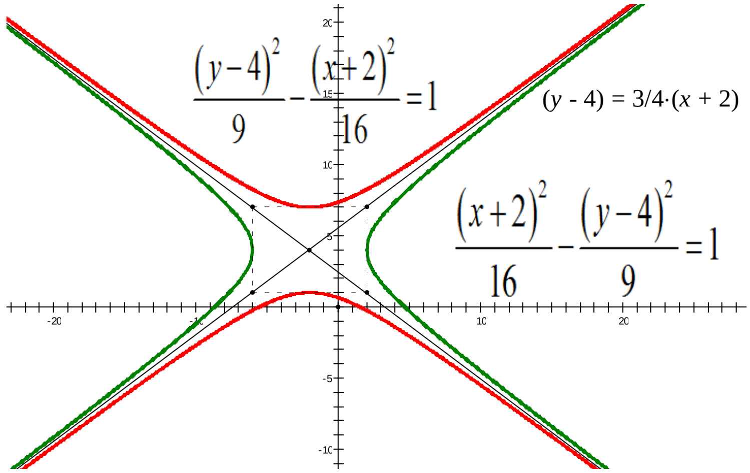
$$(y-4)^2=\frac{9\cdot 41}{16}\Rightarrow y=\underline{\mathbf{\frac{16\pm 3\sqrt{41}}{4}}}.$$

Case 2: Horizontal $(a,b)=(4,3)$

$$\frac{(x+2)^2}{16}-\frac{(y-4)^2}{9}=1$$

For $x=-7$, we have

$$(y-4)^2=9\left(\frac{25}{16}-1\right)=\frac{81}{16}\Rightarrow y=4\pm\frac{9}{4}=\underline{\mathbf{\frac{25}{4},\frac{7}{4}}}.$$



B) $4x^6-13x^4-13x^2+4=4\left(x^6+1\right)-13x^2\left(x^2+1\right)$

Using the sum of perfect cubes, $x^6+1=\left(x^2\right)^3+1^3=\left(x^2+1\right)\left(x^4-x^2+1\right)$.

$$\Leftrightarrow\left(x^2+1\right)\left(4\left(x^4-x^2+1\right)-13x^2\right)=\left(x^2+1\right)\left(4x^4-17x^2+4\right)\Leftrightarrow$$

$$\left(x^2+1\right)\left(4x^2-1\right)\left(x^2-4\right)=\underline{\left(\mathbf{x^2+1}\right)\left(\mathbf{x-2}\right)\left(\mathbf{x+2}\right)\left(\mathbf{2x-1}\right)\left(\mathbf{2x+1}\right)}$$

	4	0	−13	0	−13	0	4
2	4	8	3	6	−1	−2	0
−2	4	0	3	0	−1	0	

Alternately, dividing synthetically by $x-2$ and then by $x+2$, we get $4x^4+3x^2-1$ which factors as $\left(4x^2-1\right)\left(x^2+1\right)$ and the solution follows.

C) To insure that sin(x) is being assigned a real value, the discriminant must be nonnegative, i.e. $N\le\frac{5}{4}$.

$$\sin(x)=\frac{1\pm\sqrt{1-4(N-1)}}{2}\Rightarrow\left(2\sin(x)-1\right)^2=\left(\pm\sqrt{1-4(N-1)}\right)^2\Rightarrow$$

$4\sin^2 x-4\sin x+1=5-4N$. Transposing terms and dividing thru by 4

$\Rightarrow\ N=1+\sin x-\sin^2 x$ or $N=\cos^2 x+\sin x$

Now it was given that $x=(30k)°$ for $0\le k<12$.

$k=0, 3, 6\Rightarrow N=\underline{\mathbf{1}}$ $\qquad$ $k=1, 5\Rightarrow N=\underline{\mathbf{\frac{5}{4}}}$

$k=2, 4, 8, 10\Rightarrow N$ would be irrational $\qquad$ $k=9\Rightarrow N=\underline{\mathbf{-1}}$ $\qquad$ $k=7, 11\Rightarrow N=\underline{\mathbf{\frac{1}{4}}}$.

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Team Round - continued

D) Given: $S = x^2 + 4xy + 9y^2 + 4x + \underline{\underline{18y}} + 2017$

Recall: $(A+B+C)^2 = A^2 + B^2 + C^2 + 2AB + 2AC + 2BC$

Since there is an *xy*-term, we assume that the expression can be written in a form involving the square of a trinomial in *x*; in fact, the trinomial would have to be $(x + 2y + 2)$ to insure the correct coefficients for x^2 and xy. Proceeding with this assumption,

$(x+2y+2)^2 = \underline{x^2} + 4y^2 + 4 + \underline{4xy} + \underline{4x} + 8y$ and the underlined terms are accounted for.

The mismatched terms are $4y^2 + 8y + 4$. We need an additional $5y^2$, so we formulate an additional term of $5(y+c)^2$, where c is a constant to be determined.

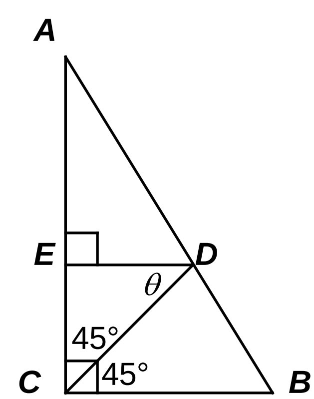
Since $5(y+c)^2 = 5y^2 + 10cy + 5c^2$, the y-coefficient will be matched if $10c + 8 = 18 \Rightarrow c = 1$.

Now $(x+2y+2)^2 + 5(y+1)^2$ matches S except for the constant and adding $\underline{2008}$ produces a perfect match.

Since $(x+2y+2)^2 + 5(y+1)^2 \ge 0$, for all real numbers x and y,

$(x+2y+2)^2 + 5(y+1)^2 + 2008$ has a *minimum* value of **<u>2008</u>** when $(x, y) =$ **<u>(0, –1)</u>**.

E) $\overline{DE} \parallel \overline{BC} \Rightarrow \theta = 45° \Rightarrow DE = EC$



As corresponding sides, $\Delta ADE \sim \Delta ABC \Rightarrow \dfrac{AE}{AC} = \dfrac{DE}{BC}$.

Switching the means in the proportion, $\dfrac{AE}{DE} = \dfrac{AC}{BC}$.

Adding "1" to both sides of the proportion,

$$\frac{AE}{DE} + \boxed{\frac{DE}{DE}} = \frac{AC}{BC} + \boxed{\frac{BC}{BC}} \Rightarrow \frac{AE+DE}{DE} = \frac{AC+BC}{BC}$$

Since $DE = EC$, we have $\dfrac{AC}{DE} = \dfrac{AC+BC}{BC}$

Cross multiplying, $DE(AC+BC) = AC \cdot BC$.

Now the magic!

Divide both sides by $AC \cdot BC \cdot DE$ and we have $\dfrac{AC+BC}{AC \cdot BC} = \dfrac{1}{BC} + \dfrac{1}{AC} = \dfrac{1}{DE}$.

Therefore, we just take the reciprocal of the given dimension! $\dfrac{2}{\sqrt{5}+1} \cdot \dfrac{\sqrt{5}-1}{\sqrt{5}-1} = \underline{\mathbf{\dfrac{\sqrt{5}-1}{2}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2017 SOLUTION KEY

Team Round - continued

F) Since $4000 = 4(1000) = 2^2 \cdot 10^3 = 2^5 \cdot 5^3$, factors which are not multiples of 5 are 1, 2, 4, 8, 16, and 32. Factors which are multiples of 5 are products of 5, 25, and 125 with any of these 6 numbers. Thus, there are 6·3 = 18 possible factors which are multiples of 5.
Clearly, no prime numbers other than 2 and 5 divide evenly into 4000. Thus, a factor of 4000 must be of the form $2^a \cdot 5^b \Rightarrow a = \mathbf{0},1,2,3,4,5$ and $b = \mathbf{0},1,2,3$, resulting in $6 \cdot 4 = 24$ total factors and $k = \dfrac{18}{24} = 75\%$.
In general, to find the number of factors of any integer:

- determine the prime factorization of the integer
- add 1 to each of the exponents
- multiply

A and *B* must be of the form $5^3 \cdot n$, where *n*, like 2^5, has 6 factors (none of which are divisible by 5).
[$\dfrac{1}{2}$ of the numbers of the form $5^1 \cdot n$, where *n* is not divisible by 5, will be divisible by 5.
$\dfrac{2}{3}$ of the numbers of the form $5^2 \cdot n$, where *n* is not divisible by 5, will be divisible by 5.
$\dfrac{3}{4}$ of the numbers of the form $5^3 \cdot n$, where *n* is not divisible by 5, will be divisible by 5 - Bingo!]
Consider *n*-values less than 32 with 6 factors (none of which are multiples of 5):
$\cancel{31}(2), \cancel{29}(2), \mathbf{28} = 2^2 \cdot 7\ (6)$ - Bingo!
Consider *n*-values greater than 32 with 6 factors (none of which are multiples of 5):
$\cancel{33}(4), \cancel{34}(4), \cancel{36}(9), \cancel{37}(2), \cancel{38}(4), \cancel{39}(4), \cancel{41}(2), \cancel{42}(8), \cancel{43}(2), \mathbf{44} = 2^2 11\ (6)$ - Bingo!
Thus, the minimum value of *B* – *A* is 125(44 –28) = 125(16) = <u>**2000**</u>.

Note: 75% of the factors of $A = 31 \cdot 5^3$ are multiples of 5 (6 out of 8, i.e., all factors except 1 and 31), but *A* does not have the same number of factors as 4000, namely 24.
Likewise, 75% of the factors of $B = 33 \cdot 5^3 = 3 \cdot 11 \cdot 5^3$ are multiples of 5 (12 out of 16, i.e. all factors except 1, 3, 11, and 33), but *B* fails to have the same total number of factors as 4000.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2017 ANSWERS

Round 1 Analytic Geometry: Anything

A) -8 B) $4\sqrt{17}$ C) $(18,6,225)$

Round 2 Alg: Factoring

A) 4 B) 40 C) $-\frac{1}{3},\frac{5}{2}$

Round 3 Trig Equations

A) 228 B) $\frac{\pi}{2},\frac{\pi}{3},\frac{2\pi}{3}$ C) $\frac{5}{7}$

Round 4 Alg 2: Quadratic Equations

A) $(1,-5,7)$ B) 79 C) $(6, -5, -69)$

Round 5 Geometry: Similarity

A) 4.8 (or $\frac{24}{5}$) B) 29 C) $\frac{25\pi}{24}$

Round 6 Alg 1: Anything

A) $-\frac{5}{2}$ B) $13_{(\text{base } 11)}$ C) $\frac{1}{12}$

Team Round

A) $\frac{7}{4}, \frac{25}{4}, \frac{16\pm 3\sqrt{41}}{4}$ D) 2008 $(0, -1)$

B) $(x^2+1)(x-2)(x+2)(2x-1)(2x+1)$ E) $\frac{\sqrt{5}-1}{2}$

C) $\pm 1,\frac{1}{4},\frac{5}{4}$ F) 2000