MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2017
ROUND 1 ALGEBRA 2: SIMULTANEOUS EQUATIONS AND DETERMINANTS
ANSWERS

A) (________ , ________ , ________)

B) ________________________________

C) __________________________ mph

A) Determine the unique ordered triple (x, y, z) which satisfies $\begin{cases} x-2y+4z=2017 \\ -x+2y=3 \\ x+y=0 \end{cases}$.

B) Compute k such that $\begin{vmatrix} 14 & 3 & -7 & 8 \\ k & 5 & 11.6 & 3 \\ -6 & 9 & 3 & 5 \\ .8 & .6 & -.4 & -.3 \end{vmatrix} = 0$.

C) Marty takes 9 minutes longer to walk a mile than Dick. Marty can walk 5 miles in the time it takes Dick to walk 8 miles. Compute Marty's walking rate in miles per hour.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2017 SOLUTION KEY

Round 1

A) Adding the first two equations, we have $4z = 2020 \Rightarrow z = 505$.
Adding the last two equations, we have $3y = 3 \Rightarrow y = 1$
$\therefore (x, y, z) = \underline{\mathbf{(-1,1,505)}}$.

B) If the entries in any two rows (or any two columns) of a matrix are proportional, then the determinant of that matrix will be zero. Comparing rows with row 2 (and columns with column 1), we note that the constants in column 1 are proportional to the corresponding constants in column 3. Specifically, multiplying the entries in column 3 by –2 produces the entries in column 1. Therefore, k = **<u>–23.2</u>** .

$$\begin{vmatrix} 14 & 3 & -7 & 8 \\ k & 5 & 11.6 & 3 \\ -6 & 9 & 3 & 5 \\ .8 & .6 & -.4 & -.3 \end{vmatrix}$$

C) Assume Dick walks a mile in T minutes and Marty takes $(T+9)$ minutes.

Rate x Time = Distance $\Rightarrow \begin{cases} (1) & R_D T = 1 \\ (2) & R_M (T+9) = 1 \end{cases}$ (Note: Rates are in miles <u>per minute</u>)

To walk 5 miles, Marty takes $\dfrac{5}{R_M}$ minutes.

To walk 8 miles, Dick takes $\dfrac{8}{R_D}$ minutes.

$\dfrac{5}{R_M} = \dfrac{8}{R_D} \Leftrightarrow R_D = \dfrac{8}{5} R_M \quad (3)$

Substituting for R_D in (1) $\Rightarrow \left(\dfrac{8}{5} R_M\right) T = 1 \Rightarrow R_M T = \dfrac{5}{8}$

Expanding (2) and substituting for $R_M T$, $\dfrac{5}{8} + 9R_M = 1 \Rightarrow 9R_M = \dfrac{3}{8} \Rightarrow R_M = \dfrac{1}{24}$ miles per minute $\Rightarrow \dfrac{1}{24} \cdot 60 = \underline{\mathbf{2.5}}$ miles per hour.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2017
ROUND 2 ALGEBRA 1: EXPONENTS AND RADICALS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) Find all nonzero integer values of k for which $8^{\frac{k}{2k-1}}$ is an integer.

B) Compute all values of x for which $\sqrt{(2x-1)(x+11)} = x+7$.

C) For positive integers x and y, $x+y=8$ and $2^{10x+y}+2^{10y+x}$ is a multiple of 10. Compute all possible values of 2^x+2^y.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2017 SOLUTION KEY

Round 2

A) For nonzero integer values of k, $\dfrac{k}{2k-1}$ produces an integer value only for $k = \underline{\mathbf{1}}$.

Now, we need to look at fractional exponents which will produce an integer result.

Since $8^{\frac{1}{3}} = 2$ and $8^{\frac{2}{3}} = 4$, we have
$$\frac{k}{2k-1} = \frac{1}{3} \Rightarrow 3k = 2k-1 \Rightarrow k = \underline{\mathbf{-1}}$$
$$\frac{k}{2k-1} = \frac{2}{3} \Rightarrow 3k = 4k-2 \Rightarrow k = \underline{\mathbf{2}}$$

B) <u>Remember</u>: $x+7 \ge 0$, since the square root on the left side of the equation must be non-negative.

Squaring both sides, $2x^2+21x-11 = x^2+14x+49 \Leftrightarrow x^2+7x-60 = (x-5)(x+12) = 0$

$\Rightarrow x = \underline{\mathbf{5}}, \cancel{-12}$.

C) $(x, y) = (7,1), (6,2), (5,3), (4,4)$

The units digits of positive integer powers of 2 are cyclic with a period of 4, i.e. they repeat in blocks of 4. $2^1, 2^2, 2^3, 2^4, 2^5, 2^6 \ldots \Rightarrow \underline{2,4,8,6,}2,4,\ldots$

The rightmost digit of $2^{71}+2^{17}$ is the same as that of $2^3+2^1 = 1\underline{0}$.

The rightmost digit of $2^{62}+2^{26}$ is the same as that of $2^2+2^2 = 0\underline{8}$.

The rightmost digit of $2^{53}+2^{35}$ is the same as that of $2^1+2^3 = 1\underline{0}$.

The rightmost digit of $2^{44}+2^{44}$ is the same as that of $2^0+2^0 = 0\underline{2}$.

Thus, $2^x+2^y = 2^7+2^1 = \underline{\mathbf{130}}$, $2^5+2^3 = \underline{\mathbf{40}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2017
ROUND 3 TRIGONOMETRY: ANYTHING

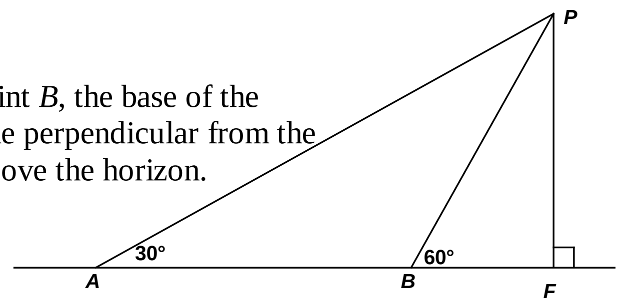
ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

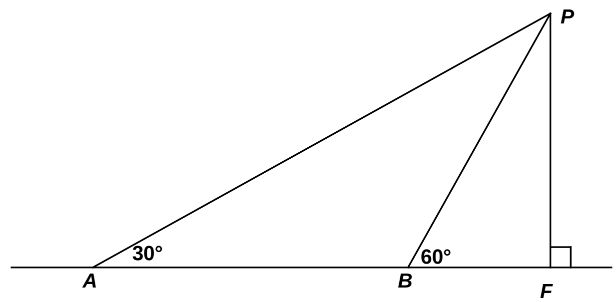
A) When a hiker walking along a level trail stops at point *A*, he notices that the peak of a mountain is 30° above the horizon. When he has walked to point *B*, the base of the mountain, he is still 10,000 feet from the foot of the perpendicular from the peak to the plane of the trail and the peak is 60° above the horizon. Compute the distance from *A* to *B*.



B) Solve for *x* over $0 \le x < \pi$. $(\tan x - i\sec x)(\tan x + i\sec x) = 7$, where $i = \sqrt{-1}$.

C) Over the interval $0 \le x < 2\pi$, the graphs of $y = \sin\left(\frac{5}{2}x\right)$ and $|3y-2| = 3$ intersect at *k* points. Compute *k*.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2017 SOLUTION KEY



Round 3

A) $BF = 10000$. Let $PF = k$ and $AB = x$.

In ΔPBF, $\tan 60° = \sqrt{3} = \dfrac{k}{10{,}000} \Rightarrow k = 10000\sqrt{3}$.

In ΔPAF,

$$\tan 30° = \frac{\cancel{\sqrt{3}}}{3} = \frac{k}{10{,}000 + x} = \frac{10{,}000\cancel{\sqrt{3}}}{10{,}000 + x} \Rightarrow x + 10{,}000 = 30{,}000 \Rightarrow x = \underline{\mathbf{20{,}000}}$$

B) $(\tan x - i\sec x)(\tan x + i\sec x) = \tan^2 x + \sec^2 x = 2\tan^2 x + 1 = 7$

$\Rightarrow \tan x = \pm\sqrt{3} \Rightarrow x = \underline{\boldsymbol{\dfrac{\pi}{3}, \dfrac{2\pi}{3}}}$

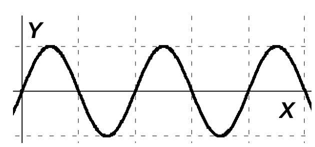
C) The graph of $y = \sin\left(\dfrac{5}{2}x\right)$ has a period of $\dfrac{2\pi}{5/2} = \dfrac{4\pi}{5}$

Thus, over the interval $0 \le x < 2\pi$, there are $\dfrac{2\pi}{\dfrac{4\pi}{5}} = 2 \cdot \dfrac{5}{4} = 2.5$ cycles of the sine function.

$|3y - 2| = 3 \Leftrightarrow 3y - 2 = \pm 3 \Leftrightarrow y = \dfrac{2 \pm 3}{3} = -\dfrac{1}{3}, \dfrac{5}{3}$

Since $-1 \le \sin\dfrac{5}{2}x \le 1$, $y = \dfrac{5}{3}$ never intersects the sine graph.

$y = -\dfrac{1}{3}$ intersects the graph of $y = \sin\left(\dfrac{5}{2}x\right)$ twice over a complete cycle.



Therefore, there are **4** points of intersection, since the last "half-cycle" is <u>above</u> the x-axis.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2017
ROUND 4 ALGEBRA 1: ANYTHING

ANSWERS

A) ______________________________

B) (_____________ , _____________)

C) ______________________________

A) Determine all ordered pairs of *positive* integers (x, y), where $y > x$, for which $x! \cdot y! = 720$.

B) 2017 is a prime number. For a unique positive integer k, $N = \sqrt{k^2 + 2017}$ is a rational number. Compute the ordered pair (k, N).

C) Compute the sum of all positive integer values of n for which the expression $\dfrac{25+3n}{2n-5}$ represents an integer.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2017 SOLUTION KEY

Round 4

A) $720 = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 2^4 3^2 5^1$
The first few factorial numbers are 1, 2, 6, 24, 120, 720.
$720 = 1 \cdot 720 = 6 \cdot 120 = 1! \cdot 6! = 3! \cdot 5! \Rightarrow (x, y) = \underline{\mathbf{(1,6),\ (3,5)}}$.

B) Squaring both sides, we have $N^2 = k^2 + 2017 \Rightarrow N^2 - k^2 = (N+k)(N-k) = 2017$.
However, since 2017 is prime, its only factors are 1 and 2017.
Therefore, $\begin{cases} N + k = 2017 \\ N - k = 1 \end{cases} \Rightarrow 2N = 2018 \Rightarrow (k, N) = \underline{\mathbf{(1008,1009)}}$

C) $\boxed{\begin{array}{r} 1 \\ 2n-5\,\overline{\big)\,3n+25} \\ \underline{2n-5} \\ n+30 \end{array}} \Rightarrow \dfrac{3n+25}{2n-5} = 1 + \dfrac{n+30}{2n-5}$

$\dfrac{n+30}{2n-5}$ is the fractional part of the mixed number equivalent of $\dfrac{3n+25}{2n-5}$.

Evaluating the fractional part, $n = 1,2,3,4,5,6 \Rightarrow \dfrac{31}{-3}, \boxed{\dfrac{32}{-1}} = -32, \boxed{\dfrac{33}{1}} = 33, \dfrac{34}{3}, \boxed{\dfrac{35}{5}} = 7, \dfrac{36}{7}$

*** As the numerator of these fractions increases by 1, the denominator increases by 2.
The positive quotients will continue to get smaller. Eventually, the denominator will exceed the numerator and the search must stop.
We will get an integer quotient when the numerator is a multiple of the denominator.

Continuing the pattern, $n = 7,8,9,...,15 \Rightarrow \dfrac{37}{9}, \dfrac{38}{11}, \boxed{\dfrac{39}{13}}, \dfrac{40}{15}, \dfrac{41}{17}, \dfrac{42}{19}, \dfrac{43}{21}, \dfrac{44}{23}, \dfrac{45}{25}$

So a 3:1 ratio was a possibility, but a 2:1 ratio was not.

Is a 1 : 1 ratio possible? $45 + k = (25 + 2k) \Leftrightarrow k = 20 \Rightarrow \dfrac{65}{65}$ or $n + 30 = 2n - 5 \Rightarrow n = 35$.

The sum of the *n*-values is 2 + 3 + 5 + 9 + 35 = **<u>54</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2017
ROUND 5 PLANE GEOMETRY: ANYTHING

ANSWERS

A) _______________________________

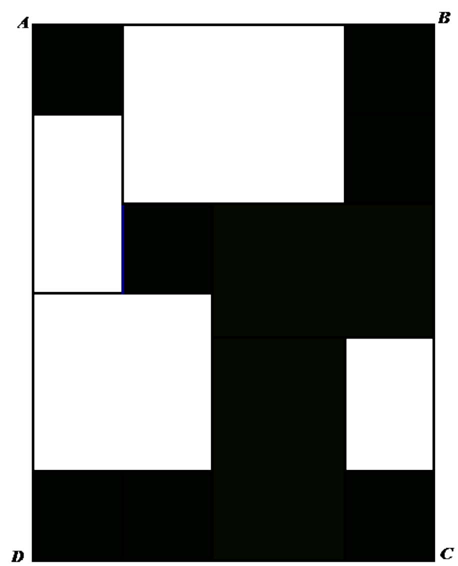
B) _______________________________

C) _______________________________

A) In ΔABC, $AB = x+3$ and $BC = 2x-1$, for positive integers x.
Compute the <u>minimum</u> value of x for which the perimeter of ΔABC is 51.

B) A 12-gon has a diagonals and an 18-gon has b diagonals. How many sides does a polygon P with $(a+b)$ diagonals have?

C) The repeating pattern in my kitchen floor tile (sans the dreadful color combination) is shown at the right. The small shaded squares inside rectangle $ABCD$ are x inches on side.
$DC = 3x + y$, $AD = 6x$, $AC = 45$ (inches)
If y is 50% more than x, compute the area (in square inches) of this rectangular pattern $ABCD$.



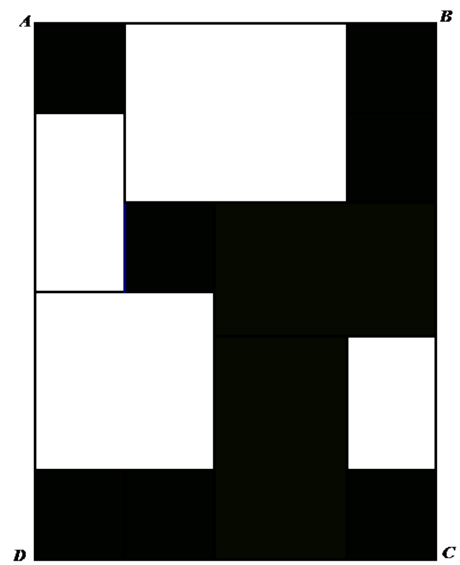
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2017 SOLUTION KEY

Round 5

A) According to the triangle inequality, $AC < AB + BC \Leftrightarrow AC < (x+3)+(2x-1) = 3x+2$.
To make x as small as possible, we must make AC as large as possible.
The maximum possible *integer* value of AC is $3x+1$.
Thus, $(x+3)+(2x-1)+(3x+1) = 51 \Rightarrow 6x = 51-3 = 48 \Rightarrow x = \underline{\mathbf{8}}$

B) $\dfrac{n(n-3)}{2} \Rightarrow$ 12-gon: 54 diagonals, 18-gon: 135 diagonals , P (n-gon):
$n(n-3) = 2(54+135) = 378$
$n = 20$ is too small, since 20(17) = 340 < 378, but $21(18) = 378$ and P has **<u>21</u>** sides.

C) Given: $DC = 3x + y$, $AD = 6x$, $AC = 45$ (inches)
$y = \dfrac{3}{2}x$
Applying the Pythagorean Theorem,
$(3x+y)^2 + (6x)^2 = 45^2$. Since $DC = 4.5x$, we have
$(4.5x)^2 + (6x)^2 = 45^2$. Multiplying through by 4,
$(9x)^2 + (12x)^2 = 4 \cdot 45^2 \Rightarrow x^2 = \dfrac{4 \cdot 45^2}{81+144} \Rightarrow x = \dfrac{2 \cdot 45}{15} = 6,\ y = 9$.
Thus, the area is $27 \cdot 36 = \underline{\mathbf{972}}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2017
ROUND 6 ALG 2: PROBABILITY AND THE BINOMIAL THEOREM

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) How many coefficients in the expansion of $(x+y)^7$ will be even?

B) An integer is randomly selected from 10 to 999, inclusive. Compute the probability that the integer is divisible by 6, given that it is divisible by 2 or 3.

C) Compute <u>all</u> possible values of k, if the sum of the three middle terms in the expansion of $\left(k+\frac{1}{k}\right)^4$ is 23.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2017 SOLUTION KEY

Round 6

A) Solution #1:

There will be a total of 8 terms, and Pascal's triangle is the fastest ways to evaluate all the coefficients. We require the 7^{th} row (which has all odd numbers) $\Rightarrow$ **<u>0</u>** (or none).

0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

Solution #2: (Annalisa Peterson - Mt. Alvernia)

To determine the number of <u>*odd*</u> numbers in row *k* of Pascal's triangle:

- Count the number of 1s in the binary representation of row number *k*.
- Raise 2 to this power.

$7 = 111_{(2)} \Rightarrow 3$ - 1s $\quad 2^3 = 8 \Rightarrow 8$ coefficients are odd

Since row *k* always contains $(k+1)$ terms, **<u>none</u>** of the coefficients are even.

B) The interval contains $999-10+1=990$ integers, 495 even and 495 odd.

Multiples of 3 range from $12 = 3\cdot 4$ to $999 = 3\cdot 333$, 330 values.

Multiples of 6 range from $12 = 6\cdot 2$ to $996 = 6\cdot 166$, 165 values.

Thus, $P(\div 6 \mid \div 2 \text{ or } 3) = \dfrac{165}{495+330-165} = \dfrac{1}{3+2-1} = \underline{\mathbf{\dfrac{1}{4}}}$.

C) The expansion is $k^4 + 4k^2 + 6 + \dfrac{4}{k^2} + \dfrac{1}{k^4}$

Solution #1 (Direct Approach)

$4k^2 + 6 + \dfrac{4}{k^2} = 23 \Rightarrow 4k^4 - 17k^2 + 4 = 0 \Leftrightarrow (4k^2-1)(k^2-4) = 0 \;\; k = \underline{\pm\mathbf{\dfrac{1}{2}},\ \pm\mathbf{2}}$.

Solution #2: (Symmetry)

$4k^2 + 6 + \dfrac{4}{k^2} = 23 \Leftrightarrow 4k^2 - 17 + \dfrac{4}{k^2} = 0 \Leftrightarrow (4k^2-1)\left(1-\dfrac{4}{k^2}\right) = 0$ and the same result follows.

Solution #3 (Even Function: $f(-x) = f(x)$ <u>and</u> symmetry)

Consider the function $f(k) = 4k^2 - 17 + \dfrac{4}{k^2}$ which is an even function.

We require that $f(k) = 0$, i.e. we are looking for the zeros of this function.

Suspecting that $\dfrac{4}{k^2}$ might be an integer, we try factors of 4. $k = 2$ works.

Since the given function is even, $k = \underline{\mathbf{\pm 2}}$. By the symmetry of the trinomial, $k = \underline{\pm\mathbf{\dfrac{1}{2}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2017
ROUND 7 TEAM QUESTIONS

ANSWERS

A) (______ , ______ , ______) D) ____________________________

B) ____________________________ E) ____________________________

C) ____________________________ F) ____________________________

A) Given: $\begin{cases} ax+by+cz=216 \\ bx+cy+az=60 \\ cx+ay+bz=84 \end{cases}$

If $a:b:c=1:2:3$ and $x+y+z=5$, compute the ordered triple (x,y,z).

B) Express the following as a simplified fraction with a <u>rationalized denominator</u>: $\dfrac{\sqrt{3+2\sqrt{2}}}{2\sqrt{1+\sqrt{2}}}$

C) Let P denote the infinite product $16cis\frac{\pi}{3}\cdot cis\frac{\pi}{6}\cdot cis\frac{\pi}{12}\cdot cis\frac{\pi}{24}\cdot \ldots$, where the first term has a coefficient of 16 and the pattern of multiplying the angle by $\frac{1}{2}$ is continued forever. $\sqrt{P}$, expressed in rectangular form, is $A + Bi$. Compute <u>all</u> possible ordered pairs (A, B).

D) 300 people were asked how they voted during an election.
3 candidates (A, B and C) were running to fill vacant positions on the Board of Selectman
The results were reported in the following curious manner:

- 54 people voted for A and B
- 66 people voted for B only
- 186 people voted for A or B, but not C
- 42 voted for A and C
- 51 voted for B and C
- 45 voted for C only

If each person voted for *at least* one of these three candidates, how many people voted for B *if and only if* they did <u>not</u> vote for C?

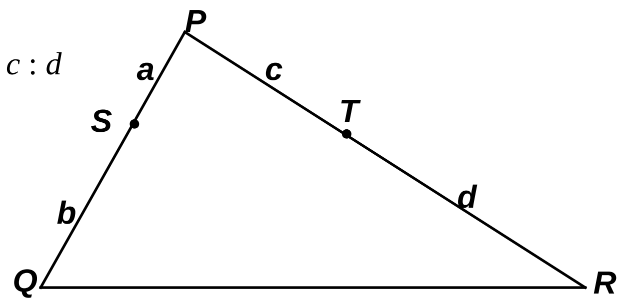
FYI:
The statement "P if and only if Q" $(P \Leftrightarrow Q)$ is a called a *bi-conditional*.

It is equivalent to "(P, if Q) or (Q, if P)", or, symbolically, $(P \Rightarrow Q) \vee (Q \Rightarrow P)$.

The bi-conditional is logically equivalent to "P and Q" or "not P and not Q".

ROUND 7 TEAM QUESTIONS - continued

E) Given: $PQ = 6$, $QR = 10$, $PS : SQ = a : b$, $PT : TR = c : d$



$$\frac{a}{b} = \frac{c}{d}, \ \frac{c}{a+b} = \frac{a}{c+d}$$

Compute the <u>maximum</u> perimeter of ΔPQR.

F) Compute the probability that between 4PM and 5PM the angle between the minute and hour hands of a clock will form an obtuse angle.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2017 SOLUTION KEY

Team Round

A) Note that the coefficients of *x*, *y*, and *z* are each $a+b+c$ when the 3 equations are added.

Therefore, $\begin{cases}(a+b+c)(x+y+z)=360\\ x+y+z=5\\ a:b:c=1:2:3\end{cases} \Rightarrow (n+2n+3n)=72 \Rightarrow n=12 \Rightarrow (a,b,c)=(12,24,36)$.

Dividing through by 12, the 3 equations become $\begin{cases}(1)\quad x+2y+3z=18\\ (2)\ 2x+3y+\ \ z=5\\ (3)\ 3x+\ \ y+2z=7\end{cases}$

Subtracting $x+y+z=5$ from each equation, $\begin{cases}(4)\quad y+2z=13\\ (5)\quad x+2y=0\\ (6)\ 2x+\ \ z=2\end{cases}$

(6) $\Rightarrow$ $z=2-2x$ (7)

Substituting for *z* in (4), $y+4-4x=13 \Rightarrow y=4x+9$ (8)

Substituting for *x* in (5), $x+2(4x+9)=0 \Rightarrow x=-2$. Substituting in (7) and (8), $(x,y,z)=\underline{(\mathbf{-2,1,6})}$.

B) Since $\left(1+\sqrt{2}\right)^2=3+2\sqrt{2}$, it follows that $\sqrt{3+2\sqrt{2}}=1+\sqrt{2}$.

Alternately, to extract the square root in the numerator, $3+2\sqrt{2}$ must be a perfect square.

So, assume there are integers *a* and *b* for which $\left(a+b\sqrt{2}\right)^2=3+2\sqrt{2}$.

Expanding and equating the rational and irrational coefficients, $\begin{cases}a^2+2b^2=3\\ 2ab=2\end{cases}$

Clearly, $(a,b)=(1,1)$ or $(-1,-1)$ satisfy both of these equations, but, since $-1-\sqrt{2}<0$ it is rejected, and we have the same result.

$\dfrac{\sqrt{3+2\sqrt{2}}}{2\sqrt{1+\sqrt{2}}}=\dfrac{1+\sqrt{2}}{2\sqrt{1+\sqrt{2}}}=\underline{\dfrac{\sqrt{\mathbf{1+\sqrt{2}}}}{\mathbf{2}}}$ The numerator $\sqrt{1+\sqrt{2}}$ cannot be further simplified.

Suppose it could.

Then we would be able to write $1+\sqrt{2}$ as $\left(A+B\sqrt{2}\right)^2$, for rational numbers *A* and *B*.

$\Rightarrow 1+\sqrt{2}=A^2+2B^2+2AB\sqrt{2} \Rightarrow \begin{cases}A^2+2B^2=1\\ 2AB=1\end{cases}$

Solving the second equation for *B* and substituting in the first, $A^2+\dfrac{1}{2A^2}=1 \Rightarrow 2A^4-2A^2+1=0$.

This equation has no real (let alone rational) roots.

Therefore, the numerator $\sqrt{1+\sqrt{2}}$ cannot be further simplified.

Team Round – continued

C) $P = 16cis\frac{\pi}{3} \cdot cis\frac{\pi}{6} \cdot cis\frac{\pi}{12} \cdot cis\frac{\pi}{24} \cdot ... = 16cis\left(\pi\left(\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+...\right)\right)$

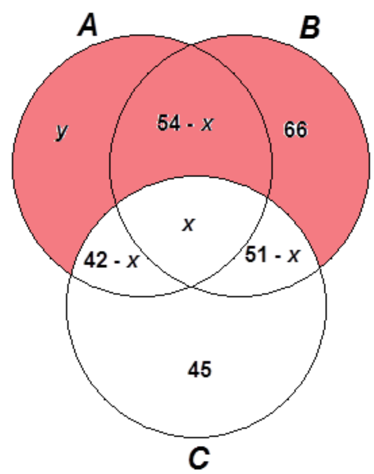
Since the argument is an infinite geometric sequence with a multiplier of $\frac{1}{2}$, it converges to

$\frac{1/3}{1-1/2} = \frac{2}{3}$ and $P = 16cis\left(\frac{2\pi}{3}+2n\pi\right)$. Thus, $\sqrt{P} = \sqrt{16}cis\left(\frac{1}{2}\cdot\left(\frac{2\pi}{3}+2n\pi\right)\right) = 4cis\left(\frac{\pi}{3}+n\pi\right)$

$n = 0 \Rightarrow 4\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right) = 4\left(\frac{1}{2}\right)+4\left(\frac{\sqrt{3}}{2}\right)i = 2+(2\sqrt{3})i$

$n = 1 \Rightarrow 4\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right) = 4\left(-\frac{1}{2}\right)+4\left(-\frac{\sqrt{3}}{2}\right)i = -2+(-2\sqrt{3})i$

$\Rightarrow (A, B) = \underline{\mathbf{(2, 2\sqrt{3})}}$, $\underline{\mathbf{(-2, -2\sqrt{3})}}$.



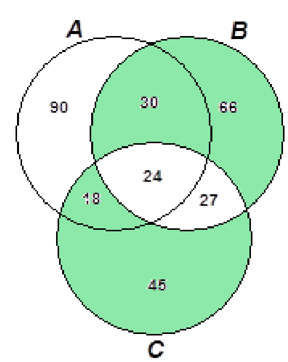
D) Given:

- 54 people voted for *A* and *B*
- 66 people voted for *B* only
- 186 people voted for *A* or *B*, but not *C*
- 42 voted for *A* and *C*
- 51 voted for *B* and *C*
- 45 voted for *C* only

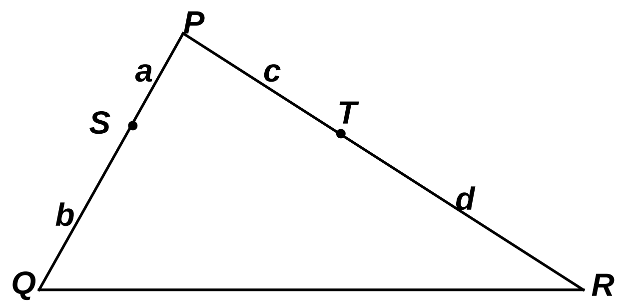
Let the Venn Diagram at the right summarize the voting.
The shaded region represents *A* or *B*, but not *C*.
Let *x* denote the people who voted for all three candidates.
Let *y* denote those people who voted for *A* only.
Since all 300 people voted for at least one candidate, the 7 disjoint regions account for all the votes.

$y+(54-x)+66 = 186 \Rightarrow y = 66+x$

Adding the *x*-expressions for all 7 regions, $324-x = 300 \Rightarrow x = 24$.
Since $P \Leftrightarrow Q$ is equivalent to $(P \text{ and } Q)$ or $(\text{not}\,P \text{ and } \text{not}\,Q)$,
the bi-conditional "voted for *B* if and only if NOT *C*" is equivalent to
"*B* and NOT *C*" or "*C* and NOT *B*".
Thus, the required regions in the Venn diagram are shaded (in green), and, we have $(30+66)+(18+45) = \underline{\mathbf{159}}$.



Team Round – continued



E)
Given: $PQ = 6$, $QR = 10$, $PS : SQ = a : b$, $PT : TR = c : d$

$$\frac{a}{b} = \frac{c}{d},\ \frac{c}{a+b} = \frac{a}{c+d}$$

Let $x = a + b$ and $y = c + d$.

$$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{x-a} = \frac{c}{y-c} \Rightarrow ay - ac = xc - ac \Rightarrow ay = xc \Rightarrow c = \frac{ay}{x}$$

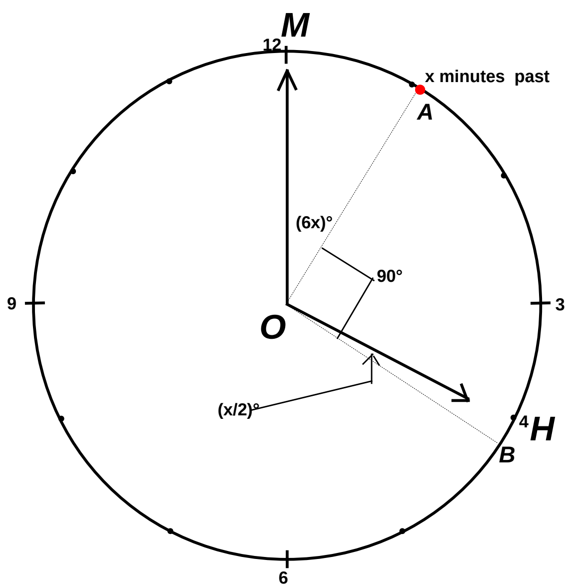
$$\frac{c}{a+b} = \frac{a}{c+d} \Rightarrow \frac{c}{x} = \frac{a}{y} = \Rightarrow ax = cy \Rightarrow ax = \left(\frac{ay}{x}\right)y = \frac{ay^2}{x} \Rightarrow x^2 = y^2 \Rightarrow x = y \Rightarrow PQ = PR$$

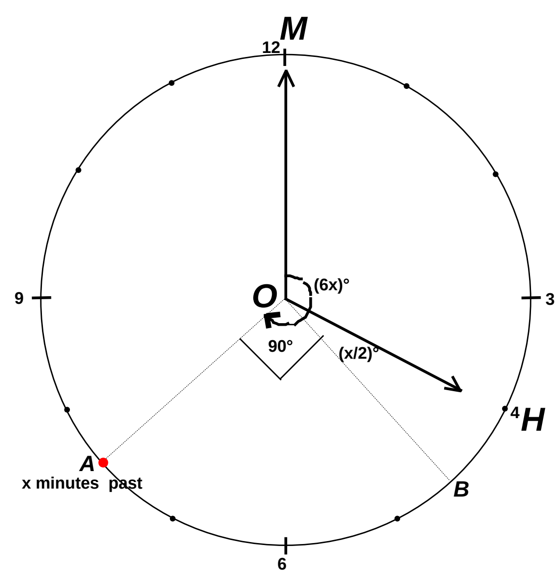
Thus, both proportions can be true if and only if ΔPQR is isosceles with base $\overline{QR}$; hence, the only possible perimeter is (6 + 6 + 10) = <u>**22**</u>.

Team Round – continued

F) Enumerating times over a continuous domain of 1 hour when the angle is obtuse requires we determine when the hands are perpendicular. At 4PM the hands form an angle of 120° and, as time passes, the angle measure first gets smaller and then it gets larger again. Note the minute makes a complete revolution every hour, while the hour hand takes 12 hours to make a complete revolution. Thus, the minute hand moves 12 times as fast as the hour hand. In one minute, the minute hand turns $\frac{1}{60}$ of a revolution, i.e. turns through $\frac{1}{60}(360)=6°$ and the hour hand turns through $\frac{1}{12}(6°)=0.5°$.

Case 1: (shortly after 4:05 PM) Case 2: (shortly after 4:35 PM)
A is the point to which the minute hand points at *x* minutes past the hour.
Not forgetting that the hour hand has also moved, we have the following diagrams:
[$m\angle MOA$ refers to an angle measured *clockwise* from *M* to *A*. On the left, it is an acute angle; on the right, it is a *reflexive* angle, i.e. an angle whose measure is greater than 180° (but less than 360°).]





From the diagram on the left, we have

$$m\angle MOA+m\angle AOB=m\angle MOH+m\angle HOB \Leftrightarrow 6x+90=120+\frac{x}{2}\Rightarrow x=\frac{60}{11}.$$

From the diagram on the right, we have

$$m\angle MOA-m\angle AOB=m\angle MOH+m\angle HOB \Leftrightarrow 6x-90=150+\frac{x}{2}\Rightarrow x=\frac{480}{11}$$

Thus, the probability is $\dfrac{\frac{60}{11}+\left(60-\frac{420}{11}\right)}{60}=\frac{1}{11}+\left(1-\frac{7}{11}\right)=\underline{\mathbf{\frac{5}{11}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2017 ANSWERS

Round 1 Algebra 2: Simultaneous Equations and Determinants

A) $(-1,1,505)$ B) -23.2 C) 2.5

Round 2 Algebra 1: Exponents and Radicals

A) -1, 1, 2 B) 5 only C) 40, 130

Round 3 Trigonometry: Anything

A) 20,000 feet B) $\frac{\pi}{3},\frac{2\pi}{3}$ C) 4

Round 4 Algebra 1: Anything

A) $(1,6),(3,5)$ B) $(1008,1009)$ C) 54

Round 5 Plane Geometry: Anything

A) 8 B) 21 C) 972

Round 6 Algebra 2: Probability and the Binomial Theorem

A) 0 (or none) B) $\frac{1}{4}$ C) $\pm 2, \pm\frac{1}{2}$

Team Round

A) $(-2,1,6)$ D) 159

B) $\frac{\sqrt{1+\sqrt{2}}}{2}$ E) 22

C) $\left(\pm 2,\pm 2\sqrt{3}\right)$ - 2 ordered pairs F) $\frac{5}{11}$